Reliability assessment of structural dynamic systems by importance sampling

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The dynamical system

$$\mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{v}}(t) + \mathbf{C}(\boldsymbol{\theta})\dot{\boldsymbol{v}}(t) + \mathbf{K}(\boldsymbol{\theta})\boldsymbol{v}(t) = \boldsymbol{D}f(t, \boldsymbol{\xi})$$

$$\boldsymbol{v}(0) = \mathbf{0}, \dot{\boldsymbol{v}}(0) = \mathbf{0}$$

$\boldsymbol{\Theta}$: random vector with PDF $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$

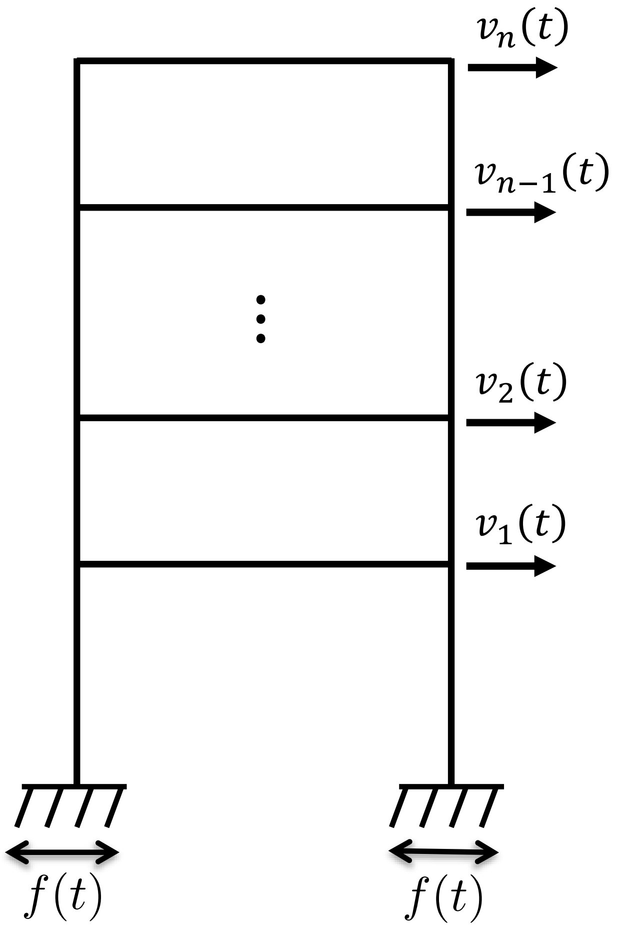
$f(t, \boldsymbol{\Xi})$: Gaussian random excitation over a time duration $[0, T]$.

At time instant t_k: $f(t_k, \boldsymbol{\Xi}) = \boldsymbol{G}_k^{\mathrm{T}}\boldsymbol{\Xi}$, where $\boldsymbol{\Xi}$ is a vector of independent standard Gaussian random variables

Linear load-response relationship:

$$h(t, \boldsymbol{\theta}, \boldsymbol{\xi}) = \int_0^t K(t-\tau, \boldsymbol{\theta}) f(\tau, \boldsymbol{\xi}) d\tau$$

$K(t, \boldsymbol{\theta})$: unit IRF of the deterministic system for $\boldsymbol{\Theta} = \boldsymbol{\theta}$



Failure event and probability of failure

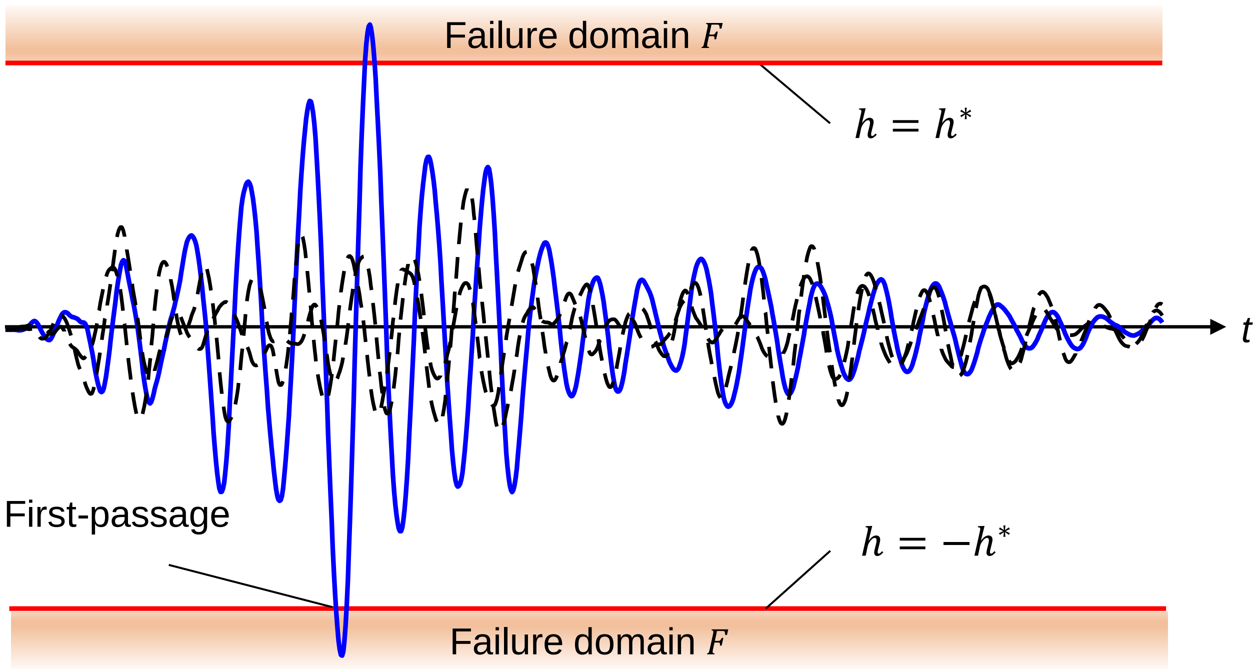
- First-passage failure:

$$F = \{\boldsymbol{\theta} \in \mathbb{R}^{n_{\theta}}, \boldsymbol{\xi} \in \mathbb{R}^{n_{\xi}} \colon \max_{0<t\leq T} |h(t, \boldsymbol{\theta}, \boldsymbol{\xi})| \geq h^*\}$$

- Probability of failure:

$$P_F = \int_{\boldsymbol{\theta}\in\mathbb{R}^{n_{\theta}}} \int_{\boldsymbol{\xi}\in\mathbb{R}^{n_{\xi}}} \mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F\}\, p_{\boldsymbol{\Xi}}(\boldsymbol{\xi}) d\boldsymbol{\xi} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

- Structural failure is a rare event
  - Standard MCS computationally intractable
  - Advanced MCS/ variance reduction techniques required



Importance sampling

- Consider IS probability density $q_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta},\boldsymbol{\xi})$

$$P_F = \int_{\boldsymbol{\theta}\in\mathrm{R}^{n_{\theta}}} \int_{\boldsymbol{\xi}\in\mathrm{R}^{n_{\xi}}} \mathrm{I}\{(\boldsymbol{\theta},\boldsymbol{\xi}) \in F\} \frac{p_{\boldsymbol{\Xi}}(\boldsymbol{\xi}) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{q_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta},\boldsymbol{\xi})} q_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta},\boldsymbol{\xi}) d\boldsymbol{\xi} d\boldsymbol{\theta}$$

- IS estimate of P_F:

$$\hat{P}_F = \frac{1}{N} \sum_{i=1}^{N} \mathrm{I}\{(\boldsymbol{\theta}^i,\boldsymbol{\xi}^i) \in F\} \frac{p_{\boldsymbol{\Xi}}(\boldsymbol{\xi}^i) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}^i)}{q_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta}^i,\boldsymbol{\xi}^i)}$$

where $\{(\boldsymbol{\theta}^i,\boldsymbol{\xi}^i), i = 1, \ldots, N\}$ are i.i.d. samples from $q_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta}^i,\boldsymbol{\xi}^i)$

- Select: $q_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta},\boldsymbol{\xi}) = q_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) q_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})$ with
  - $q_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})$ according to [Au & Beck 2001]
  - $q_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ by cross entropy (CE) optimization [Kanjilal et. al. 2021]

IS density $q_{\Xi}(\xi|\theta)$ for conditional FPP

- Conditional FPP by importance sampling:

$$P_{F|\Theta}(\boldsymbol{\theta}) = \int_{\xi \in \mathrm{R}^{n_\xi}} \mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F\} \frac{p_{\Xi}(\boldsymbol{\xi})}{q_{\Xi}(\boldsymbol{\xi}|\boldsymbol{\theta})} q_{\Xi}(\boldsymbol{\xi}|\, \boldsymbol{\theta}) d\boldsymbol{\xi}$$

Time instants $\{t_1, \dots, t_{n_T}\}$ in $[0, T]$

$$F(\boldsymbol{\theta}) = \cup_{k=1}^{n_T} F_k(\boldsymbol{\theta})$$, where

$$F_k(\boldsymbol{\theta}) = \{\boldsymbol{\xi} \in \mathbb{R}^{n_\xi} : |h(t_k, \boldsymbol{\theta}, \boldsymbol{\xi})| \geq h^*\}$$

- A very efficient IS density to [Au & Beck 2001] :

$$q_{\Xi}(\boldsymbol{\xi}|\boldsymbol{\theta}) = \sum_{k=1}^{n_T} w_k(\boldsymbol{\theta}) \frac{\mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F_k(\boldsymbol{\theta})\} p_{\Xi}(\boldsymbol{\xi})}{\mathrm{Prob}[F_k(\boldsymbol{\theta})]}$$

  - Normalized weights $w_k(\boldsymbol{\theta}) = \dfrac{\mathrm{Prob}[F_k(\boldsymbol{\theta})]}{\sum_{l=1}^{n_T} \mathrm{Prob}[F_l(\boldsymbol{\theta})]}$

Conditional FPP by importance sampling

- Conditional FPP by importance sampling:

$$P_{F|\Theta}(\boldsymbol{\theta}) = \tilde{P}(\boldsymbol{\theta}) \int_{\xi \in \mathrm{R}^{n_\xi}} \frac{1}{\sum_{k=1}^{n_T} \mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F_k(\boldsymbol{\theta})\}} q_\Xi(\boldsymbol{\xi} |\, \boldsymbol{\theta}) d\boldsymbol{\xi}$$

where $\tilde{P}(\boldsymbol{\theta}) = \sum_{k=1}^{n_T} \mathrm{Prob}[F_k(\boldsymbol{\theta})]$

Time instants $\{t_1, \ldots, t_{n_T}\}$ in $[0, T]$

$$h(t_k, \boldsymbol{\theta}, \boldsymbol{\Xi}) = \boldsymbol{r}_k^T(\boldsymbol{\theta})\boldsymbol{\Xi}$$

where $\boldsymbol{r}_k^T(\boldsymbol{\theta}) = \Delta t \sum_{s=1}^{k} c_s K(t_k - t_s, \boldsymbol{\theta}) \boldsymbol{G}_s^T$

- Instantaneous failure probabilities:
  - $F_k(\boldsymbol{\theta}) = F_k^+(\boldsymbol{\theta}) + F_k^-(\boldsymbol{\theta})$ (mutually exclusive events)
  - $\mathrm{Prob}[F_k^+(\boldsymbol{\theta})] = \mathrm{Prob}[F_k^-(\boldsymbol{\theta})] = \Phi\left(-\frac{h^*}{\|\boldsymbol{r}_k(\boldsymbol{\theta})\|}\right)$

Estimation of the unconditional FPP

- Direct approach computationally infeasible
- Adopt importance sampling

$$P_F = \int_{\boldsymbol{\theta}\in\mathbb{R}^{n_{\boldsymbol{\theta}}}} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

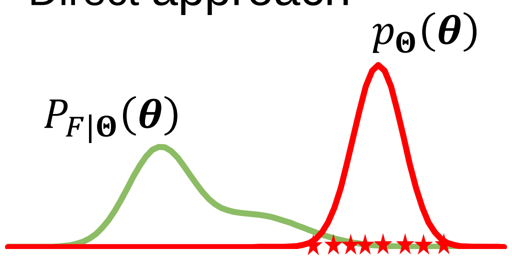
$$\approx \sum_{i=1}^{M} P_{F|\boldsymbol{\Theta}}\left(\boldsymbol{\theta}^i\right) \frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}^i)}{q_{\boldsymbol{\Theta}}(\boldsymbol{\theta}^i)}$$

$\{\boldsymbol{\theta}^i, i = 1, \ldots, M\}$ are independent samples $\sim q_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$

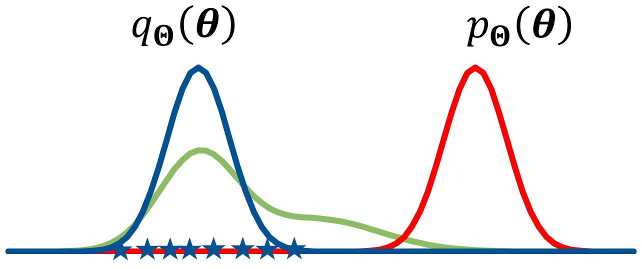
- Optimal IS density: $q_{\boldsymbol{\Theta}}^*(\boldsymbol{\theta}) = \frac{1}{P_F} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$
- **Proposed approach:**

  Obtain approximation of $q_{\boldsymbol{\Theta}}^*(\boldsymbol{\theta})$ by cross entropy method

Direct approach



Importance sampling

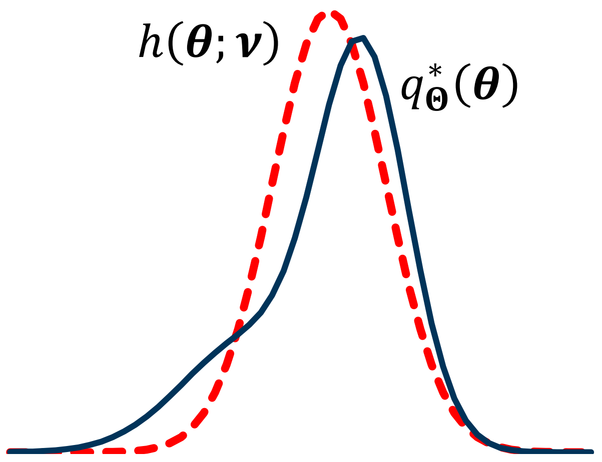


Determination of $q_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ by CE method

- Select parametric density family $h(\boldsymbol{\theta};\boldsymbol{v}), \boldsymbol{v} \in \mathcal{V}$
  - $h(\boldsymbol{\theta};\ \boldsymbol{v}_0)$ corresponds to $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ for $\boldsymbol{v}_0 \in \mathcal{V}$
  - find $\boldsymbol{v}^*$ such that $h(\boldsymbol{\theta};\boldsymbol{v}^*) \approx q^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$

- Determine $\boldsymbol{v}^*$ by CE optimization
  - $$\boldsymbol{v}^* = \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmin}}\, D_{KL}\left(q^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta}), h(\boldsymbol{\theta};\boldsymbol{a})\right)$$
    $$= \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmax}}\ \mathrm{E}_{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}\left[P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})\ln\left(h(\boldsymbol{\theta};\boldsymbol{a})\right)\right]$$

- CE optimization can be solved using samples from $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$
  - can be *computationally expensive*
  - adopt a *multi-level approach*



$$D_{KL}\left(q^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta}), h(\boldsymbol{\theta};\boldsymbol{v})\right)$$

Kullback-Leibler divergence

Information lost in representing $q^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ by the parametric density $h(\boldsymbol{\theta};\boldsymbol{v})$.

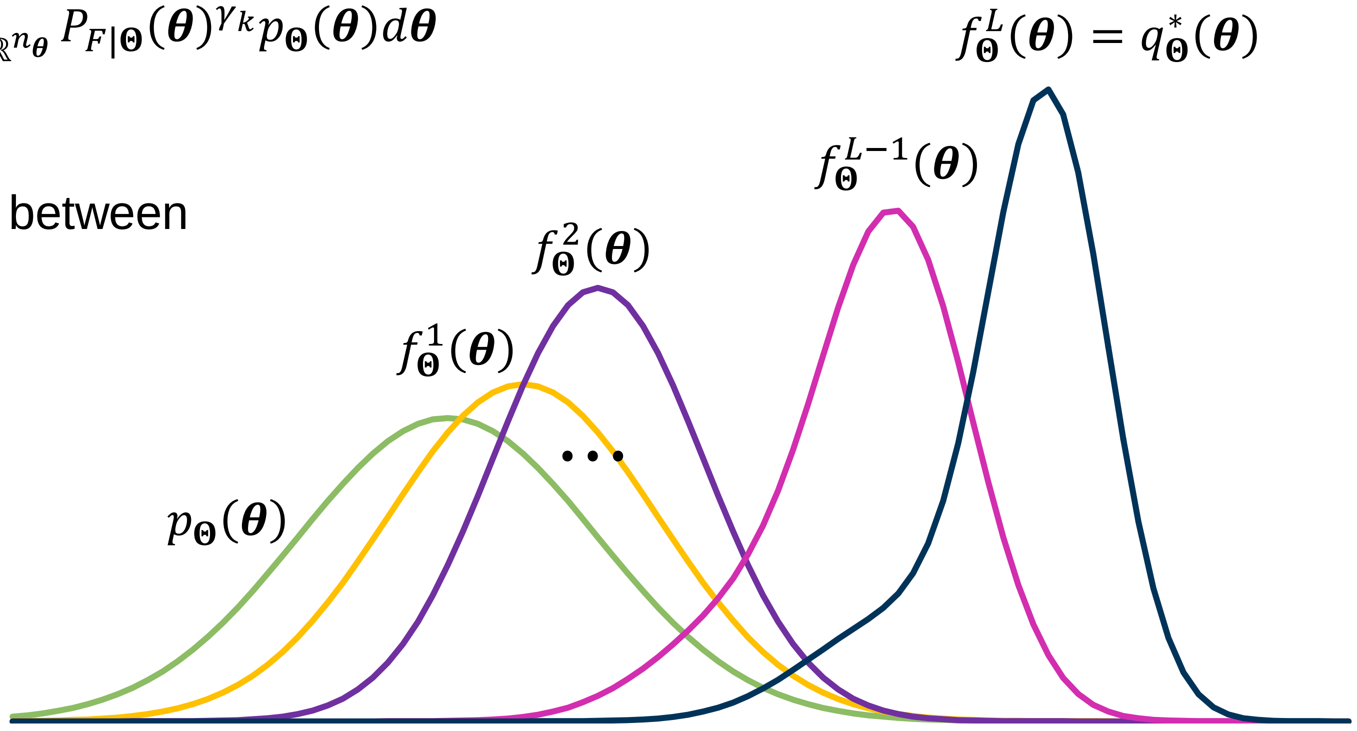
Multi-level CE method for $q_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$: intermediate densities

Sequence of intermediate target densities

$$f_{\boldsymbol{\Theta}}^{k}(\boldsymbol{\theta}) = \frac{1}{C_k} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$$

with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_L = 1$ and $C_k = \int_{\boldsymbol{\theta} \in \mathbb{R}^{n_{\boldsymbol{\theta}}}} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$

- Tempering parameters control the difference between successive densities
- Smooth transition between $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and $q_{\boldsymbol{\Theta}}^{*}(\boldsymbol{\theta})$



CE optimization in the intermediate levels

Sequential approach based on importance sampling

Approximate $f_{\boldsymbol{\Theta}}^{k}(\boldsymbol{\theta})$ by $h(\boldsymbol{\theta};\boldsymbol{v}_k)$

$$\boldsymbol{v}_k = \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmin}}\, D_{KL}\left(f_{\boldsymbol{\Theta}}^{k}(\boldsymbol{\theta}), h(\boldsymbol{\theta};\boldsymbol{a})\right)$$

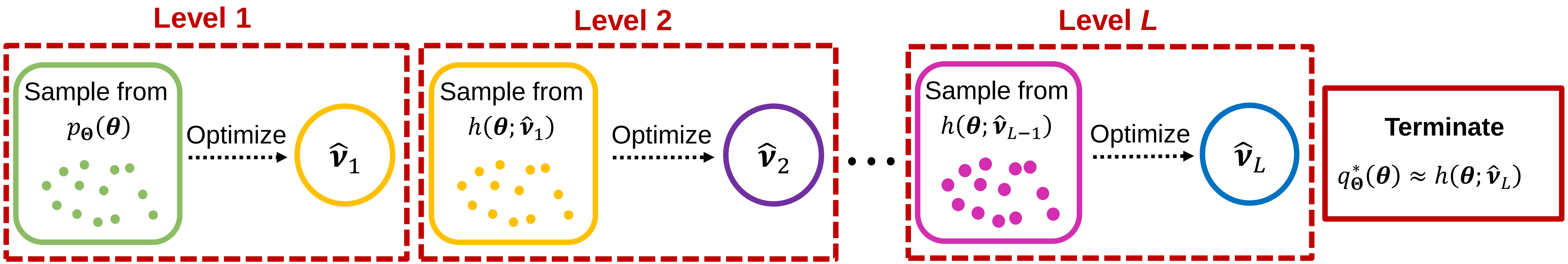
$$= \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmax}}\ \mathrm{E}_{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}\left[P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k}\ln\left(h(\boldsymbol{\theta};\boldsymbol{a})\right)\right]$$

$$= \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmax}}\ \mathrm{E}_{h(\boldsymbol{\theta};\widehat{\boldsymbol{v}}_{k-1})}\left[P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k}\frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{h(\boldsymbol{\theta};\widehat{\boldsymbol{v}}_{k-1})}\ln\left(h(\boldsymbol{\theta};\boldsymbol{a})\right)\right]$$

$\widetilde{W}_k(\boldsymbol{\theta};\widehat{\boldsymbol{v}}_{k-1})$

$$\widehat{\boldsymbol{v}}_k = \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmax}}\ \sum_{i=1}^{N}\frac{P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}^i)^{\gamma_k}p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}^i)}{h(\boldsymbol{\theta}^i;\widehat{\boldsymbol{v}}_{k-1})}\ln\left(h(\boldsymbol{\theta}^i;\boldsymbol{a})\right)$$

$\{\boldsymbol{\theta}^i, i=1,\dots,N\}$ generated from $h(\boldsymbol{\theta};\widehat{\boldsymbol{v}}_{k-1})$



Adaptive selection of the levels

Adaptive selection of γ_k

$$\gamma_k = \underset{\gamma \in (\gamma_{k-1}, 1)}{\operatorname{argmin}} \left(\hat{\delta}_{\widetilde{W}_k}(\gamma) - \delta_{target}\right)^2$$

$\hat{\delta}_{\widetilde{W}_k}(\gamma)$: sample CoV of $\{\widetilde{W}_k(\boldsymbol{\theta}^i; \hat{\boldsymbol{v}}_{k-1}), i = 1, \dots N\}$

- Ensures that $f_{\boldsymbol{\Theta}}^k(\boldsymbol{\theta})$ is approximated well with a small number of samples from $h(\boldsymbol{\theta}; \hat{\boldsymbol{v}}_{k-1})$.
- Ensures that the number of effective samples used to fit the parametric model takes a target value:

  $\mathrm{ESS} = N / \left(1 + \hat{\delta}_{\widetilde{W}_k}^2(\gamma)\right)$
- Equivalent to bounding the sample CoV of the IS estimate of the normalizing constant C_k
- We select $\delta_{target} = 1.5$. This leads to $\delta_{\hat{C}_k} \approx 0.05$ for $N = 1000$.
- Convergence after L steps when $\gamma_L = 1$. Consider $q_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = h(\boldsymbol{\theta}; \hat{\boldsymbol{v}}_L) \approx q_{\boldsymbol{\Theta}}^*(\boldsymbol{\theta})$.

Evaluation of $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ during CE optimization

- The CE optimization procedure requires repeated evaluation of $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$
- By the Poisson Approximation [Rice 1944]:

$$P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) \approx 1 - \exp\left(-\int_0^T \alpha(t; h^*, \boldsymbol{\theta})\, dt\right)$$

$\alpha(t; h^*, \boldsymbol{\theta})$**:** out-crossing rate at time t

- Ensures smooth convergence of the CE method

Estimator for FPP

- Exact failure probability by importance sampling :

$$P_F = \int_{\boldsymbol{\theta}\in\mathrm{R}^{n_{\boldsymbol{\theta}}}} \int_{\boldsymbol{\xi}\in\mathrm{R}^{n_{\xi}}} \frac{\tilde{P}(\boldsymbol{\theta})}{\sum_{k=1}^{n_T} \mathrm{I}\{(\boldsymbol{\theta},\boldsymbol{\xi}) \in F_k(\boldsymbol{\theta})\}} W(\boldsymbol{\theta}) q_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta},\boldsymbol{\xi}) d\boldsymbol{\xi} d\boldsymbol{\theta}$$

where $q_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta},\boldsymbol{\xi}) = q_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) q_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = q_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) h(\boldsymbol{\theta};\hat{\boldsymbol{v}}_L)$ and $W(\boldsymbol{\theta}) = p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})/h(\boldsymbol{\theta};\hat{\boldsymbol{v}}_L)$

- IS estimator

$$\hat{P}_F = \frac{1}{N_{IS}} \sum_{i=1}^{N_{IS}} \frac{\tilde{P}(\boldsymbol{\theta}^i)}{\sum_{k=1}^{n_T} \mathrm{I}\{(\boldsymbol{\theta}^i,\boldsymbol{\xi}^i) \in F_k(\boldsymbol{\theta}^i)\}} W(\boldsymbol{\theta}^i)$$

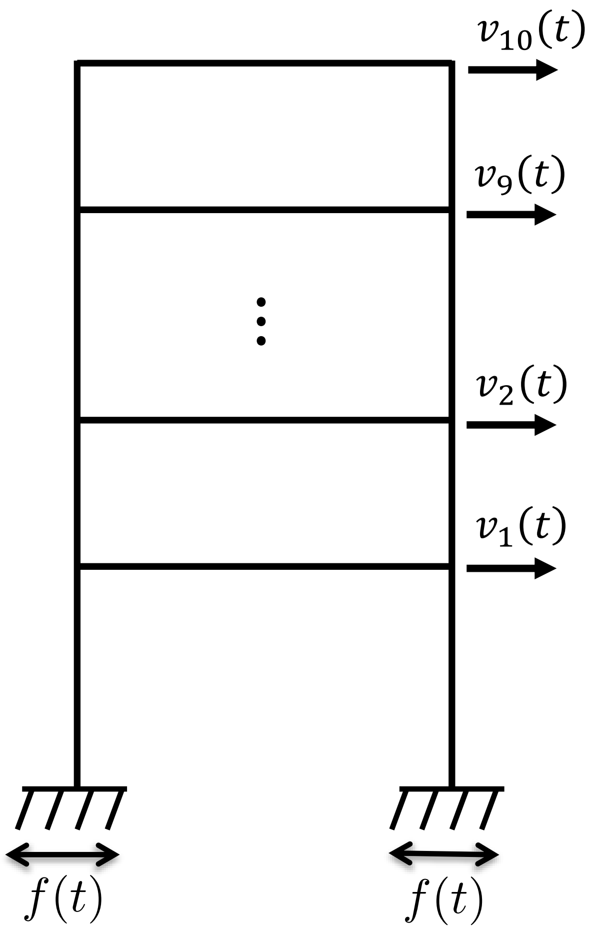
where $\{(\boldsymbol{\theta}^i,\boldsymbol{\xi}^i); i = 1,\dots,N_{IS}\}$ are independent samples from $q_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) h(\boldsymbol{\theta},\hat{\boldsymbol{v}}_L)$

Numerical example: 10-story linear frame

Dynamical system:

$$\mathbf{M}\ddot{\boldsymbol{v}}(t) + \mathbf{C}\dot{\boldsymbol{v}}(t) + \mathbf{K}\boldsymbol{v}(t) = [m_1; \dots; m_{10}]f(t)$$

- $\boldsymbol{v}(t) = [v_1(t); \dots; v_{10}(t)]$: displacement vector
- $f(t)$: filtered modulated Gaussian white noise of duration $T = 20s$
- $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ defined in terms of lumped masses $\{m_1, \dots, m_{10}\}$, inter-story stiffness $\{k_1, \dots, k_{10}\}$ and damping ratios $\{\eta_1, \dots, \eta_{10}\}$.
- Critical response $h(t, \boldsymbol{\theta}, \boldsymbol{\xi}) = v_{10}(t) - v_9(t)$
- First-passage probability: $\Pr\left(\max_{0<t\leq T} h(t, \boldsymbol{\theta}, \boldsymbol{\xi}) \geq h^*\right)$



Selection of the parametric family $h(\theta; v)$

- Choose a family that includes the nominal density $p_{\mathbf{\Theta}}(\boldsymbol{\theta})$
- Apply iso-probabilistic transformation s.t. $\mathbf{\Theta} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
- Multivariate normal distribution
  - Analytical solution of CE optimization problem [Rubinstein & Kroese 2004]
  - # of parameters: $\frac{n_{\boldsymbol{\theta}}(n_{\boldsymbol{\theta}}+3)}{2}$
- von Mises-Fisher-Nakagami (vMFN) distribution [Papaioannou et al. 2019]
  - Approximate analytical updating rules are available
  - # of parameters: $n_{\boldsymbol{\theta}} + 3$
- Mixture models can be applied for representing complex failure domains

Case 1: 10 random variables

- Random parameter vector: $\boldsymbol{\Theta} = [k_1; \dots; k_{10}]$ – components are independent truncated Gaussian random variables
- Threshold: $h^* = 0.01\text{m}$
- Two variants of the estimator considered :
    - non-adaptive case in which we take $N_{IS} = N$
    - adaptive case in which N_{IS} is adapted on the fly to ensure that $\hat{\delta}_{\hat{P}_F} \leq \delta^*$

Case 1: Non-adaptive case of the IS estimator

Reference solution obtained by standard MCS with 10^6 samples is 1.27×10^{-3}

N	CEIS-mvn-fixN			CEIS-vMFN-fixN		
	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_T	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_T
125	2.21×10^{-3}	7.135	644 (519+125)	1.21×10^{-3}	0.205	404 (279+125)
250	1.18×10^{-3}	0.237	1135 (885+250)	1.22×10^{-3}	0.111	783 (533+250)
500	1.23×10^{-3}	0.104	1755 (1255+500)	1.21×10^{-3}	0.063	1540 (1040+500)
1000	1.25×10^{-3}	0.053	3090 (2090+1000)	1.22×10^{-3}	0.054	3000 (2000+1000)

- N_T is higher for CEIS-mvn-fixN: CE optimization requires more steps to converge
- CEIS-vMFN-fixN shows superior performance in terms of sample CoV of the estimates
- Performance gap reduces with increase in the number of samples per level
- The poor performance of CEIS-mvn-fixN is due to the large number of parameters. For small *N*, the available number of effective samples is inadequate.

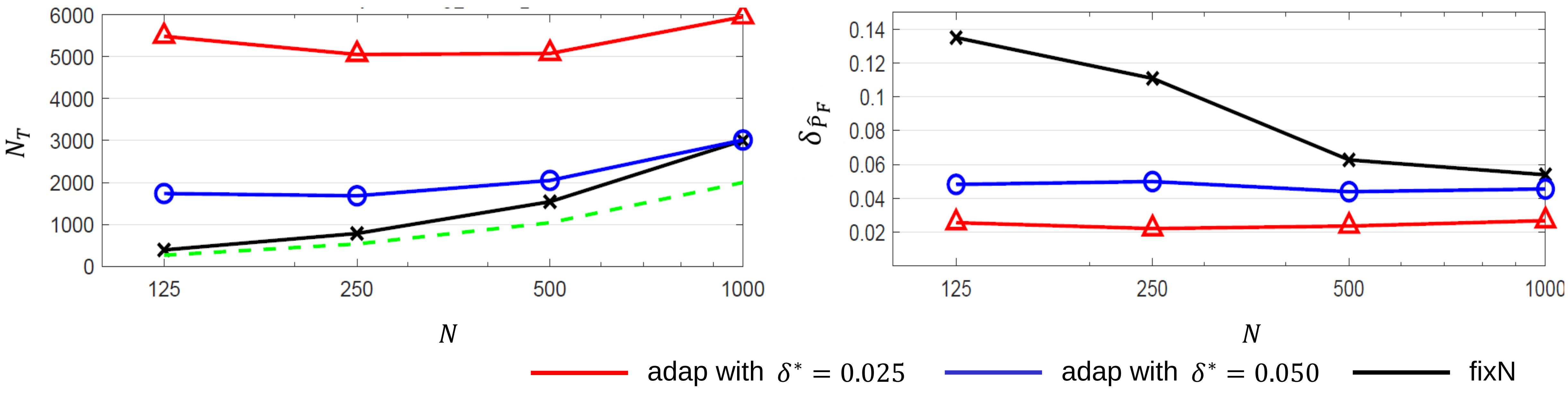
Case 1: Adaptive case of the IS estimator$(\delta^* = 0.05)$

Reference solution obtained by standard MCS with 10^6 samples is 1.27×10^{-3}

N	CEIS-mvn-adap			CEIS-vMFN-adap		
	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_T	$\hat{P}_F$	$\delta_{\hat{P}_F}$	N_T
250	-	-	-	1.22×10^{-3}	0.044	1682 (543+1139)
500	1.23×10^{-3}	0.049	2722 (1250+1472)	1.22×10^{-3}	0.050	2048 (1020+1028)
1000	1.22×10^{-3}	0.051	3190 (2080+1110)	1.22×10^{-3}	0.046	3012 (2000+1012)

- CEIS-mvn-adap requires larger number of samples to converge in the reliability estimation step
- Sample mean and CoV of the estimates with the two parametric densities are comparable

Case 1: Influence of number of samples, *N* (vMFN)



- CEIS-vMFN-fixN: Monotonic increase in N_T and decrease in $\delta_{\hat{P}_F}$ with increase in N (expected)
- CEIS-vMFN-adap: sample CoV remains close to the prescribed thresholds
- CEIS-vMFN-adap: for $N \geq 250$, no significant benefit in terms of the number of samples required in the reliability estimation step
- Adaptive variant of the IS estimator requires lesser number of samples to yield sample CoV $\approx 5\%$.

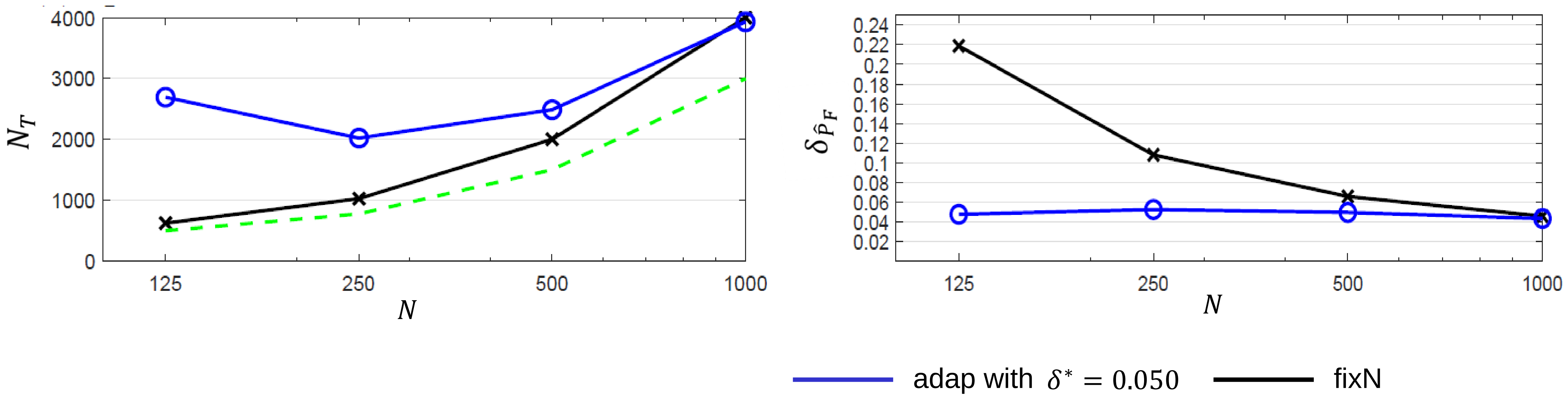
Case 2: 30 random variables

- Random parameter vector: $\boldsymbol{\Theta} = [m_1; \dots; m_{10}; k_1; \dots; k_{10}; \eta_1; \dots; \eta_{10}]$ – components are independent truncated Gaussian random variables [Schuëller & Pradlwarter 2007].
- Threshold: $h^* = 0.013\text{m}$
- CEIS implemented with vMFN model. Results correspond to $N = 250$ and $\delta^* = 0.05$.

	DMC	CEIS-fixN	CEIS-adap	LS	SS-MCMC	SS-Hybrid
$\hat{P}_F$	5.20×10^{-5}	5.21×10^{-5}	5.22×10^{-3}	6.0×10^{-5}	6.60×10^{-5}	5.90×10^{-5}
$\delta_{\hat{P}_F}$	0.023	0.108	0.053	0.120	0.580	0.460
N_T	3.50×10^7	1030	$2024\ (792+1232)$	360	2300	2645

- CEIS demonstrates significantly superior performance compared to SS-MCMC and SS-Hybrid
- Estimates from LS have smaller variability and require less computational effort
- Superior performance of LS comes at the expense of reduced robustness

Case 2: Influence of the number of samples in the CE method



- Even with $N = 125$, CEIS-vMFN-fixN and CEIS-vMFN-adap outperform SS-MCMC and SS-hybrid
- Adaptive variant of the IS estimator requires lesser number of samples to yield sample CoV $\approx 5\%$.

Summary

- Adaptive importance sampling strategy to estimate first-passage probability of uncertain linear structures.
- Effective IS density of the uncertain parameters accomplished through the multi-level CE method.
- Optimal IS density of the uncertain parameters approached by approximating a sequence of intermediate target densities.
- Density sequence constructed by introducing smoothening of the conditional FPP.
- Two variants of the IS estimator employed. If there is a target CoV of the FPP, adaptive case is computationally more efficient.
- Two parametric probability density models are employed.
- Proposed approach is a black-box method and outperforms other sampling-based methods for the problem.

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