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Reliability sensitivity analysis with FORM

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Joint work with Daniel Straub

The reliability problem

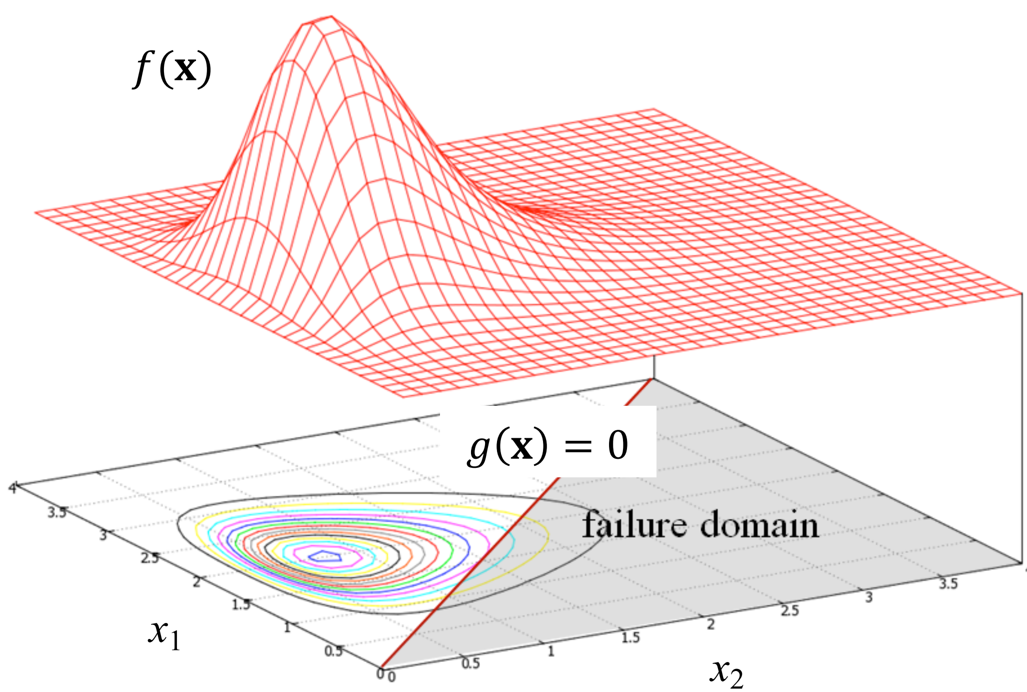
Input random variables: $\mathbf{X} = [X_1; X_2; \dots; X_n]$ with joint PDF $f(\mathbf{x})$

Assume that the components of $\mathbf{X}$ are independent, i.e., $f(\mathbf{x}) = \prod_{i=1}^{n} f_i(x_i)$

Limit-state function $g(\mathbf{x})$; Failure event $F = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 0\}$

Probability of failure: $p_F = \Pr(F)$

$$p_F = \Pr(F) = \int_{g(\mathbf{x}) \leq 0} f(\mathbf{x}) d\mathbf{x}$$



The reliability problem (II)

Probability of failure:

$$p_F = \Pr(F) = \int_{g(\mathbf{x})\leq 0} f(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^n} I(g(\mathbf{x}) \leq 0)f(\mathbf{x})d\mathbf{x} = \mathrm{E}[I(g(\mathbf{X}) \leq 0)]$$

where:

$$I(g(\mathbf{x}) \leq 0) = \begin{cases} 1, & \text{if } g(\mathbf{x}) \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

Reliability sensitivity analysis

- Gradient-based sensitivity analysis: How does a change in the (deterministic) input parameters influences p_F?
  - Gradient-based measures
  - Distribution parameter sensitivities: $\frac{\partial p_F}{\partial \mu}$, $\frac{\partial p_F}{\partial \sigma}$ [Wu 1994]
  - Sensitivities with respect to a limit-state parameter θ: $\frac{\partial p_F}{\partial \theta}$
    [Jensen et al. 2009; Papaioannou et al. 2013, 2018]

Reliability sensitivity analysis (II)

- Variance-based/global sensitivity analysis: How does the variability of the input random variables influences p_F?
  - Based on the variance decomposition of a related function of the input uncertainties $\mathbf{X}$, $Q = h(\mathbf{X})$
  - $Q = I(g(\mathbf{X}) \leq 0)$ [Li et al. 2012; Wei et al. 2012]
  - $Q = \mathrm{E}_{\mathbf{X}_A}[I(g(\mathbf{X}_A, \mathbf{X}_B) \leq 0)|\mathbf{X}_B]$ [Wang et al. 2013; Ehre et al. 2018]

Reliability sensitivity analysis (III)

- Decision-theoretic sensitivity analysis: How does the reduction of uncertainty of the input random variables influence the optimality of an engineering decision?
    - Based on decision theory – the concept of Value of Information (VoI) [Felli & Hazen 1998; Straub 2014]
    - Decision theoretic reliability sensitivities apply the VoI concept to decisions related to safety and reliability-based design [Straub et al. 2021]

Variance-based reliability sensitivity analysis

How does the variability of the input random variables influences p_F?

Variance-based sensitivity analysis

Consider an n-dimensional independent random vector $\mathbf{X}$ and a function $Q = h(\mathbf{X})$.

ANOVA representation:

$$h(\mathbf{X}) = h_0 + \sum_{i=1}^{n} h_i(X_i) + \sum_{i<j}^{n} h_{ij}\left(X_i, X_j\right) + \cdots + h_{1\ldots n}(X_1, \ldots, X_n)$$

with $h_0 = \mathrm{E}[Q]$ and $\mathrm{E}\left[h_{\mathbf{v}}(\mathbf{X}_{\mathbf{v}}) | \mathbf{X}_{\mathbf{v}\setminus i}\right] = 0$ for any subset $\mathbf{X}_{\mathbf{v}} \subseteq \mathbf{X}$ and any $i \in \mathbf{v}$.

Variance decomposition:

$$\mathrm{Var}(Q) = \sum_{i=1}^{n} V_i + \sum_{i<j}^{n} V_{ij} + \cdots + V_{1\ldots n}$$

where

$$V_i = \mathrm{Var}(\mathrm{E}[Q|X_i])$$

$$V_{ij} = \mathrm{Var}\left(\mathrm{E}\left[Q\middle|X_i, X_j\right]\right) - V_i - V_j$$

and so on

Variance-based sensitivity indices

First-order (Sobol') indices:

$$S_i = \frac{V_i}{\mathrm{Var}(Q)}$$

Second-order indices:

$$S_{ij} = \frac{V_{ij}}{\mathrm{Var}(Q)}$$

and so on

It is:

$$\sum_{i=1}^{n} S_i + \sum_{i<j}^{n} S_{ij} + \cdots + S_{1\ldots n} = 1$$

Variance-based sensitivity indices (II)

Total effect indices

$$S_i^T = \frac{\mathrm{E}[\mathrm{Var}(Q|\mathbf{X}_{\sim i})]}{\mathrm{Var}(Q)} = 1 - \frac{\mathrm{Var}(\mathrm{E}[Q|\mathbf{X}_{\sim i}])}{\mathrm{Var}(Q)}$$

It is:

$$\sum_{i=1}^{n} S_i^T \geq 1$$

Variance-based reliability sensitivities

Failure event $F = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 0\}$

Quantity of interest: $Z = I(g(\mathbf{X}) \leq 0)$

It is $\mathrm{E}[Z] = p_F, \mathrm{Var}[Z] = p_F(1 - p_F)$

First-order indices:

$$S_{F,i} = \frac{\mathrm{Var}(\mathrm{E}[Z|X_i])}{\mathrm{Var}(Z)} = \frac{\mathrm{Var}(\Pr[F|X_i])}{p_F(1 - p_F)}$$

Note: The F-O index is equivalent to the moment-independent importance measure [Li et al. 2012]

$$\delta_i^p = \mathrm{E}[(\Pr[F] - \Pr[F|X_i])^2] = \mathrm{Var}(\mathrm{E}[Z|X_i])$$

Evaluation of $S_{F,i}$ can be performed by a variety of sampling or approximation methods [Wei et al. 2012, Li et al. 2012, Perrin & Defaux 2019, Li et al. 2019]

Variance-based reliability sensitivities (II)

Total-effect indices:

$$S_{F,i}^{T} = 1 - \frac{\mathrm{Var}(\mathrm{E}[Z|\mathbf{X}_{\sim i}])}{\mathrm{Var}(Z)} = 1 - \frac{\mathrm{Var}(\Pr[F|\mathbf{X}_{\sim i}])}{p_F(1-p_F)}$$

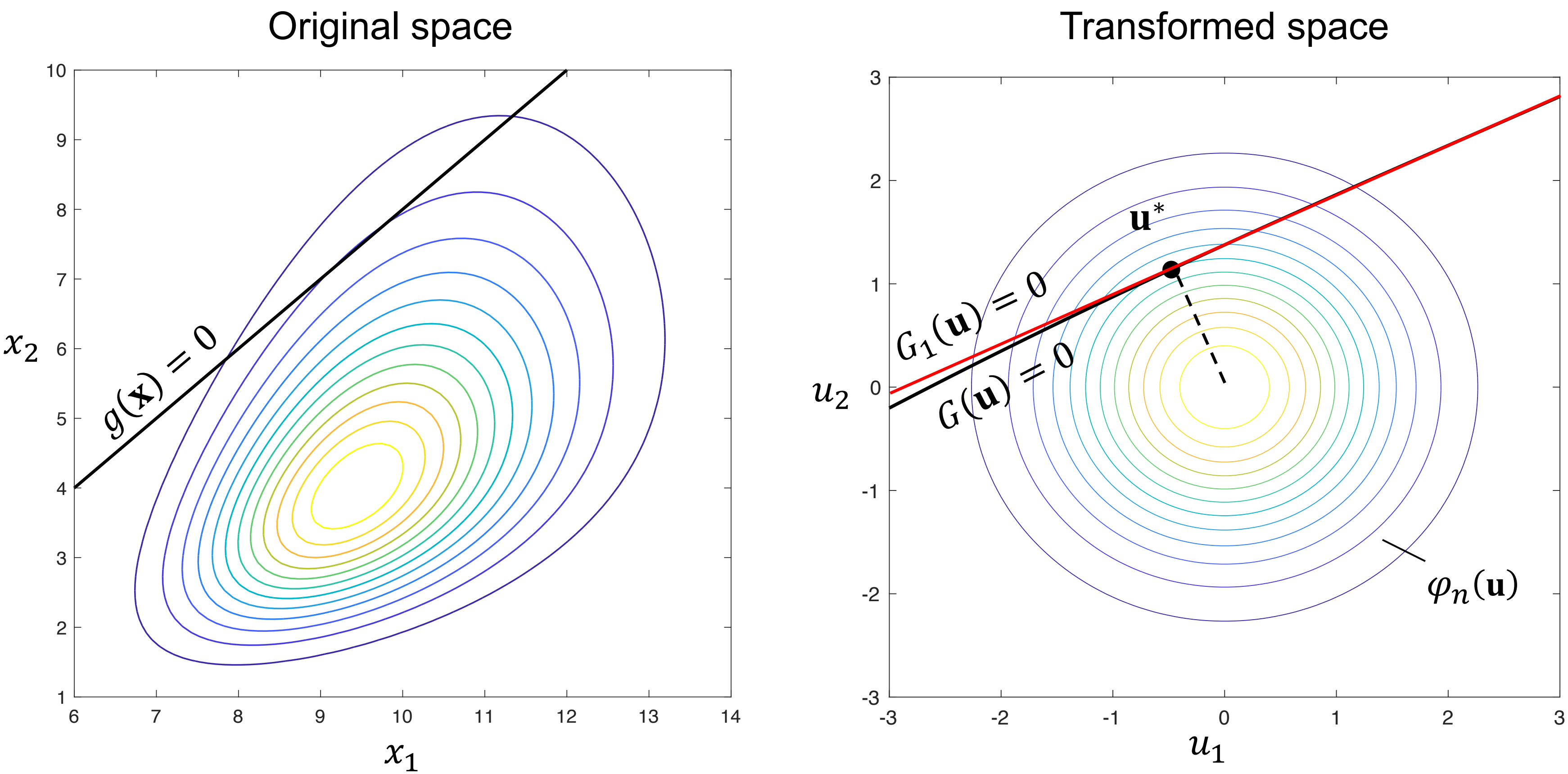
Evaluation of $S_{F,i}^{T}$ can be performed by sampling-based methods [Wei et al. 2012]

Notes

- First-order indices can be used for factor prioritization, to determine which random variable if learned will increase the accuracy of p_F the most
- Total-effect indices can be used for factor fixing, to determine the random variables with $S_{F,i}^{T} \approx 0$, which if fixed will not impact the prediction of p_F

First order reliability method (FORM)

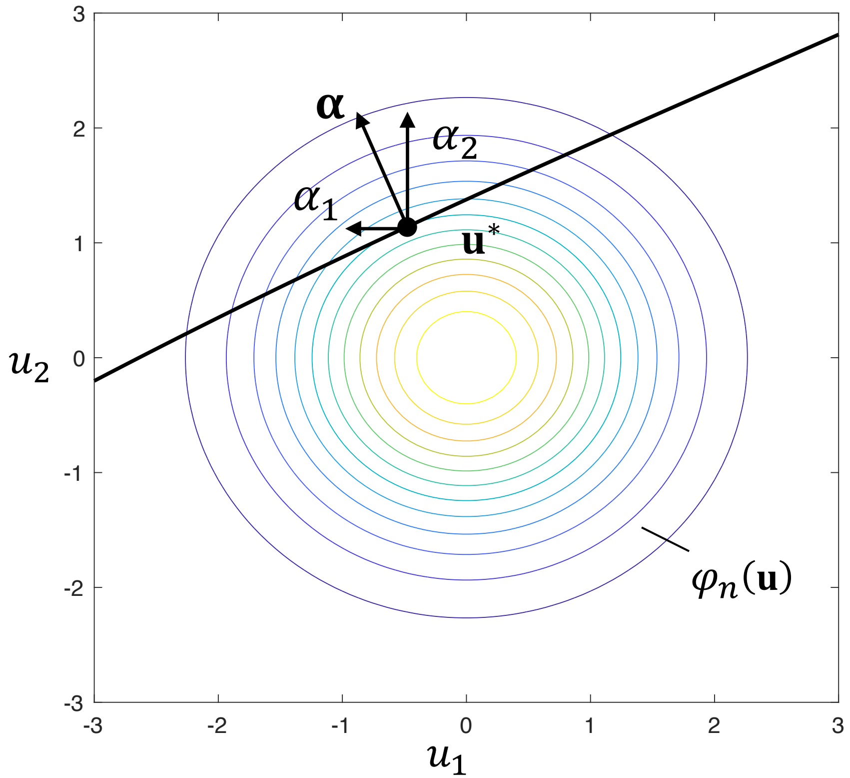
- Transform $\mathbf{X}$ to an equivalent space $\mathbf{U} \sim N(\mathbf{0}, \mathbf{I})$; Transformation operator: $\mathbf{U} = \mathbf{T}(\mathbf{X})$
- Transformed limit-state function: $G(\mathbf{u}) = g\left[\mathbf{T}^{-1}(\mathbf{u})\right]$
- Choose $\mathbf{u}^*$ as the most likely (probable) point on the hypersurface $G(\mathbf{u}) = 0$



The FORM α-factors

Normalized negative gradient of the limit-state function at the design point $\mathbf{u}^*$

$$\boldsymbol{\alpha} = -\frac{\nabla G(\mathbf{u}^*)}{\|\nabla G(\mathbf{u}^*)\|} = \frac{\mathbf{u}^*}{\beta}$$



FORM approximation

Approximation of limit-state function in $\mathbf{U}$-space

$$G(\mathbf{U}) \approx G_1(\mathbf{U}) = \nabla G(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*) = \|\nabla G(\mathbf{u}^*)\|(\beta - \boldsymbol{\alpha}\mathbf{U})$$

where $\beta = \boldsymbol{\alpha}\mathbf{u}^*$ is the FORM reliability index

FORM approximation of the failure domain:

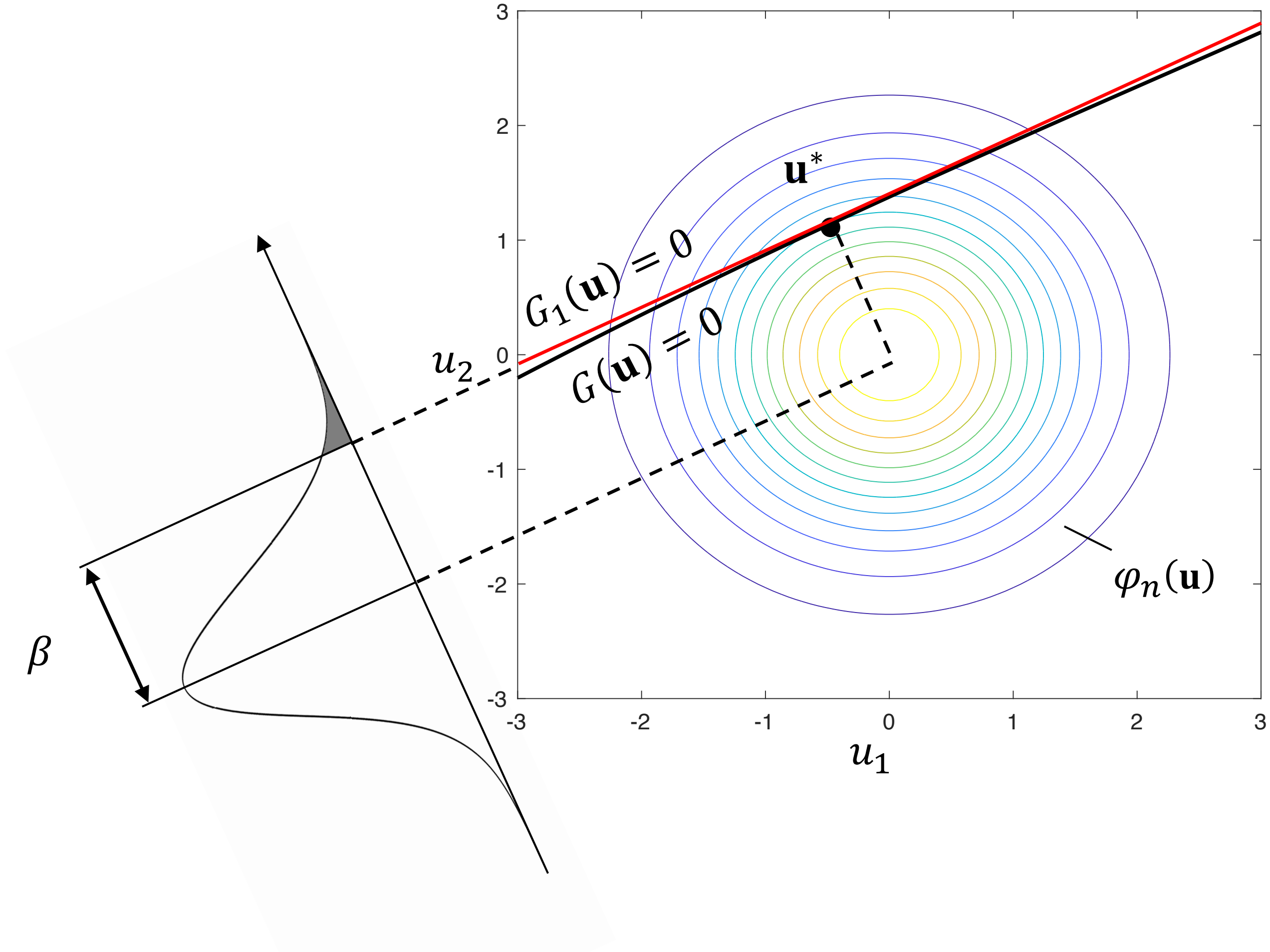
$$F \approx F_1 = \{\mathbf{u} \in \mathbb{R}^n : G_1(\mathbf{u}) \leq 0\} = \{\mathbf{u} \in \mathbb{R}^n : \boldsymbol{\alpha}\mathbf{U} \geq \beta\}$$

FORM approximation of the probability of failure:

$$p_F \approx p_{F_1} = \Pr(\boldsymbol{\alpha}\mathbf{U} \geq \beta) = \Phi(-\beta)$$

FORM approximation

Probability approximation: $p_F \approx p_{F_1} = \Pr(\boldsymbol{\alpha}\mathbf{U} \geq \beta) = \Phi(-\beta)$



FORM α-factors as variance-based sensitivities

Variance decomposition of FORM approximation of the linearized limit-state function $G_1(\mathbf{U}) = \|\nabla G(\mathbf{u}^*)\|(\beta - \boldsymbol{\alpha}\mathbf{U})$:

$$\mathrm{Var}(G_1) = \|\nabla G(\mathbf{u}^*)\|^2 \sum_{i=1}^{n} \alpha_i^2 = \|\nabla G(\mathbf{u}^*)\|^2$$

First-order indices of G_1

$$S_{G_1,i} = \frac{\mathrm{Var}(\mathrm{E}[G_1|U_i])}{\mathrm{Var}(G_1)} = \frac{\|\nabla G(\mathbf{u}^*)\|^2 \alpha_i^2}{\|\nabla G(\mathbf{u}^*)\|^2} = \alpha_i^2$$

Variance-based SA with FORM

Quantity of interest: $Z_1 = I(G_1(\mathbf{U}) \leq 0)$

It is $\mathrm{E}[Z_1] = p_{F_1}$,$\mathrm{Var}[Z_1] = p_{F_1}(1 - p_{F_1})$

First-order indices:

$$S_{F_1,i} = \frac{\mathrm{Var}(\mathrm{E}[Z_1|U_i])}{\mathrm{Var}(Z_1)} = \frac{\mathrm{Var}(\Pr[F_1|U_i])}{p_{F_1}\left(1 - p_{F_1}\right)}$$

It is

$$\Pr[F_1|U_i = u_i] = \Pr\left[\sum_{j=1,j\neq i}^{n} \alpha_j U_i \geq \beta - \alpha_i u_i\right] = \Phi\left(\frac{\alpha_i u_i - \beta}{\sqrt{\sum_{j=1,j\neq i}^{n} \alpha_j^2}}\right) = \Phi\left(\frac{\alpha_i u_i - \beta}{\sqrt{1 - \alpha_i^2}}\right)$$

Variance-based SA with FORM

and

$$\mathrm{Var}(\Pr[F_1|U_i]) = \mathrm{Var}\left(\Phi\left(\frac{\alpha_i U_i - \beta}{\sqrt{1-\alpha_i^2}}\right)\right) = \cdots = \int_0^{\alpha_i^2} \varphi_2(-\beta, -\beta, r)\,dr$$

with

$$\varphi_2(-\beta, -\beta, r) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\frac{\beta^2}{1+r}\right)$$

Variance-based SA with FORM

Second-order index:

$$S_{F_1,ij} = \frac{V_{F_1,ij}}{p_{F_1}\left(1 - p_{F_1}\right)}$$

with

$$V_{F_1,ij} = \mathrm{Var}\left(\Pr\left[F_1 \middle| U_i, U_j\right]\right) - V_{F_1,i} - V_{F_1,j}$$

It is

$$\mathrm{Var}\left(\Pr\left[F_1 \middle| U_i, U_j\right]\right) = \int_0^{\alpha_i^2+\alpha_j^2} \varphi_2(-\beta, -\beta, r)dr$$

Variance-based SA with FORM

Total effect index

$$S_{F_1,i}^T = 1 - \frac{\mathrm{Var}(\Pr[F_1|\mathbf{U}_{\sim i}])}{p_{F_1}\left(1 - p_{F_1}\right)}$$

It is

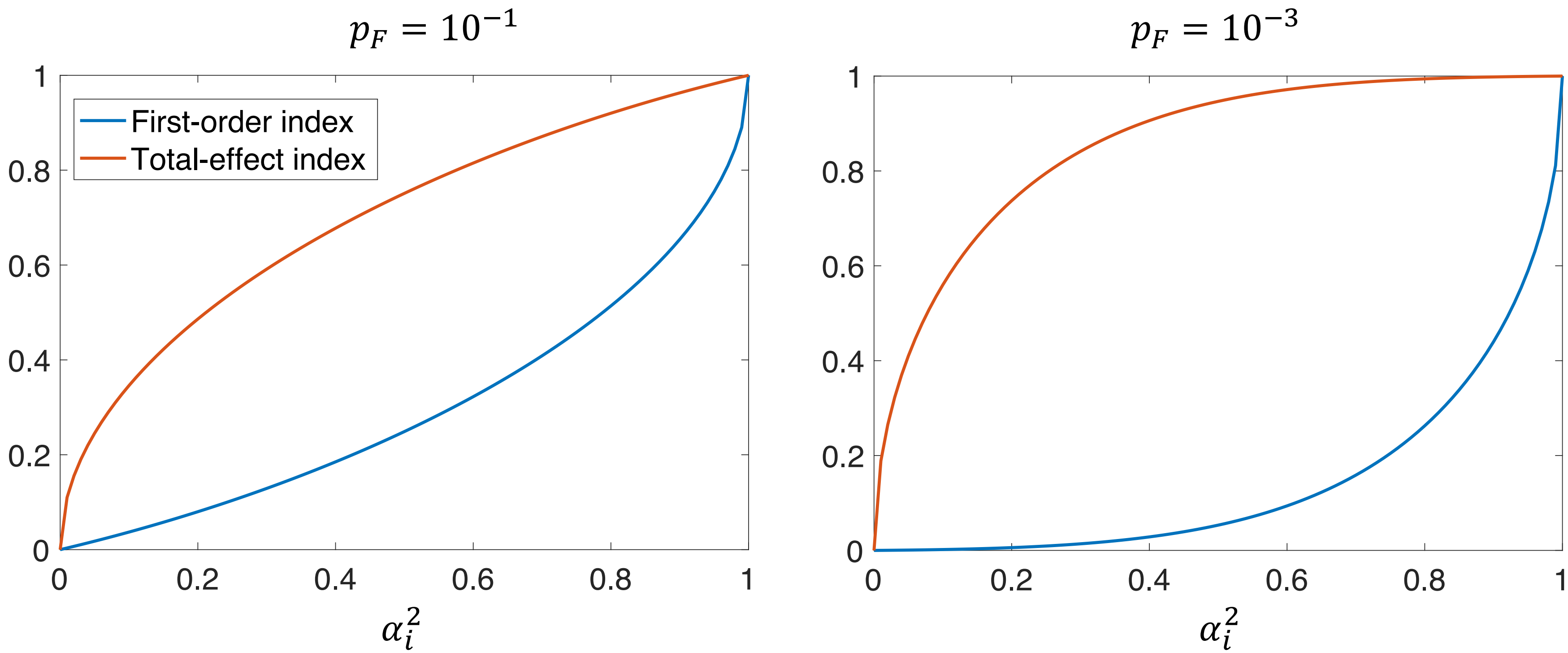
$$\mathrm{Var}(\Pr[F_1|\mathbf{U}_{\sim i}]) = \int_0^{1-\alpha_i^2} \varphi_2(-\beta, -\beta, r)dr$$

Therefore

$$S_{F_1,i}^T = 1 - \frac{1}{p_{F_1}\left(1 - p_{F_1}\right)} \int_0^{1-\alpha_i^2} \varphi_2(-\beta, -\beta, r)dr = \frac{1}{p_{F_1}\left(1 - p_{F_1}\right)} \int_{1-\alpha_i^2}^{1} \varphi_2(-\beta, -\beta, r)dr$$

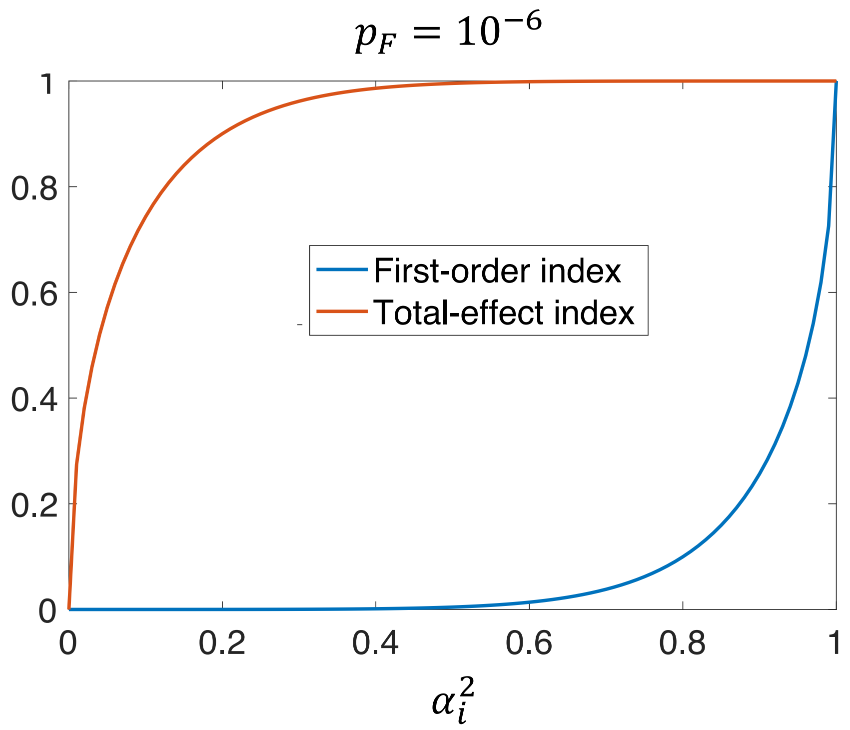
Illustration

FORM-based first-order and total effect indices vs. α_i^2 for varying p_F values



Illustration

FORM-based first-order and total effect indices vs. α_i^2 for varying p_F values



Observations

- The ranking of the FORM α-factors is consistent with the one of both the first-order and total-effect indices of the indicator function of the FORM approximation
- The indices $S_{F_1,i}$ take much smaller values than the indices $S^T_{F_1,i}$
- The difference between $S_{F_1,i}$ and $S^T_{F_1,i}$ increases with decrease of p_{F_1}

Decision-theoretic reliability sensitivity analysis

How does the reduction of uncertainty of the input random variables influence the optimality of an engineering decision?

Value of information

Consider a set of decision alternatives $\mathcal{A}$ related to an engineering system.

Optimal decision:

$$a_{opt} = \arg\min_{a \in \mathcal{A}} \mathrm{E}[L(\mathbf{X}, a)]$$

Assume that data $\mathbf{d}$ is available that can be used to update the distribution of $\mathbf{X}$ with Bayesian analysis.

The a-posteriori optimal decision given $\mathbf{d}$ is

$$a_{opt|\mathbf{d}} = \arg\min_{a \in \mathcal{A}} \mathrm{E}[L(\mathbf{X}, a)|\mathbf{d}]$$

Conditional value of information:

$$CVOI(\mathbf{d}) = \mathrm{E}\left[L\left(\mathbf{X}, a_{opt}\right)|\mathbf{d}\right] - \mathrm{E}\left[L\left(\mathbf{X}, a_{opt|\mathbf{d}}\right)|\mathbf{d}\right]$$

(Expected) value of information:

$$EVOI = \mathrm{E}_{\mathbf{d}}[CVOI(\mathbf{d})]$$

Decision-theoretic sensitivity analysis

Assume that variable X_i is known to take value x_i. The optimal decision becomes:

$$a_{opt|x_i} = \arg\min_{a \in \mathcal{A}} \mathrm{E}[L(\mathbf{X}, a)|X_i = x_i]$$

Conditional value of partial perfect information (CVPPI):

$$CVPPI_i(x_i) = \mathrm{E}\left[L\left(\mathbf{X}, a_{opt}\right)|X_i = x_i\right] - \mathrm{E}\left[L\left(\mathbf{X}, a_{opt|x_i}\right)|X_i = x_i\right]$$

Expected value of partial perfect information (EVPPI):

$$EVPPI_i = \mathrm{E}[CVPPI_i(X_i)] = \mathrm{E}\left[L\left(\mathbf{X}, a_{opt}\right)\right] - \mathrm{E}\left[L\left(\mathbf{X}, a_{opt|x_i}\right)\right]$$

Relation of EVPPI to Sobol' indices

Consider the quadratic loss function:

$$L(\mathbf{X}, a) = (h(\mathbf{X}) - a)^2$$

Then

$$a_{opt} = \arg\min_{a \in \mathcal{A}} \mathrm{E}[L(\mathbf{X}, a)] = \mathrm{E}[h(\mathbf{X})]$$

$$a_{opt|x_i} = \arg\min_{a \in \mathcal{A}} \mathrm{E}[L(\mathbf{X}, a)|X_i = x_i] = \mathrm{E}[h(\mathbf{X})|X_i = x_i]$$

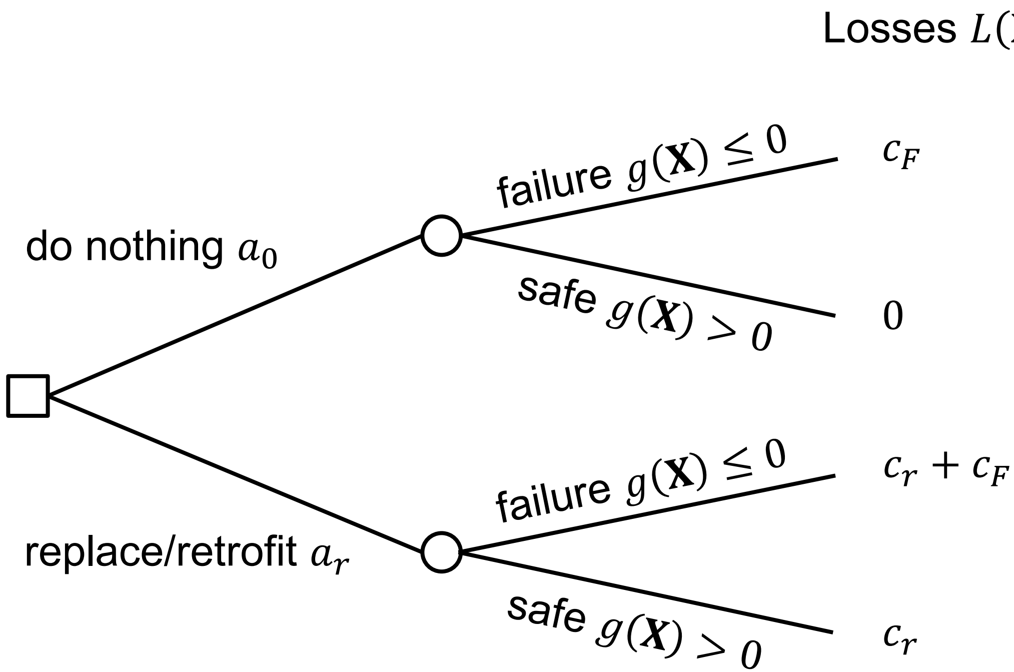
The EVPPI reads

$$EVPPI_i = \mathrm{Var}\big(h(\mathbf{X})\big) - \mathrm{E}[\mathrm{Var}(h(\mathbf{X})|X_i)] = \mathrm{Var}(\mathrm{E}[h(\mathbf{X})|X_i])$$

Dividing with $\mathrm{Var}\big(h(\mathbf{X})\big)$, we get

$$\frac{EVPPI_i}{\mathrm{Var}\big(h(\mathbf{X})\big)} = S_i$$

Safety assessment – decision tree



Assuming that $\Pr(F|a_r) \ll \Pr(F) = p_F$, we have:

$$\mathrm{E}[L(\mathbf{X}, a)] = \begin{cases} c_F p_F, & a = a_0 \\ c_r, & a = a_r \end{cases}$$

Safety assessment – decision analysis

A-priori optimal decision:

$$a_{opt} = \arg \min_{a \in \{a_0, a_r\}} \mathrm{E}[L(\mathbf{X}, a)] = \begin{cases} a_0, & p_F \leq \dfrac{c_r}{c_F} \\ a_r, & \text{else} \end{cases}$$

Assume that variable X_i is known to take value x_i. The optimal decision becomes:

$$a_{opt|x_i} = \arg \min_{a \in \{a_0, a_r\}} \mathrm{E}[L(\mathbf{X}, a) | X_i = x_i] = \begin{cases} a_0, & p_F(x_i) \leq \dfrac{c_r}{c_F} \\ a_r, & \text{else} \end{cases}$$

where $p_F(x_i) = \Pr(F | X_i = x_i)$.

Safety assessment – EVPPI

Conditional value of partial perfect information (CVPPI):

$$CVPPI_i(x_i) = \mathrm{E}\left[L\left(\mathbf{X}, a_{opt}\right)|X_i = x_i\right] - \mathrm{E}\left[L\left(\mathbf{X}, a_{opt|x_i}\right)|X_i = x_i\right]$$

The CVPPI is nonzero only if the optimal decision is changed, i.e.,

$$CVPPI_{F,i}(x_i) = \begin{cases} |c_F p_F(x_i) - c_r|, & a_{opt|x_i} \neq a_{opt} \\ 0, & a_{opt|x_i} = a_{opt} \end{cases}$$

The EVPPI reads:

$$EVPPI_{F,i} = \mathrm{E}\left[CVPPI_{F,i}(X_i)\right] = \int_{\Omega_{X_i}} |c_F p_F(x_i) - c_r| f_i(x_i) dx_i$$

where $\Omega_{X_i} = \left\{x_i \in \mathbb{R} : a_{opt|x_i} \neq a_{opt}\right\}$

Evaluation of $p_F(x_i)$, and, hence, of $EVPPI_i$ can be performed by the sampling approach of [Li et al. 2019]

Decision-theoretic SA with FORM

FORM approximation: $p_F \approx p_{F_1} = \Pr(\boldsymbol{\alpha}\mathbf{U} \geq \beta) = \Phi(-\beta)$ with $\mathbf{U} \sim N(\mathbf{0}, \mathbf{I})$

It is

$$p_{F_1}(u_i) = \Pr[F_1 | U_i = u_i] = \Phi\left(\frac{\alpha_i u_i - \beta}{\sqrt{1 - \alpha_i^2}}\right)$$

FORM approximation of EVPPI for safety assessment:

$$EVPPI_{F,i} \approx EVPPI_{F_1,i} = \int_{\Omega_{U_i}} \left| c_F p_{F_1}(u_i) - c_r \right| \varphi(u_i) du_i$$

where $\Omega_{U_i} = \left\{u_i \in \mathbb{R} : a_{opt|u_i} \neq a_{opt}\right\}$, i.e., the collection of all u_i for which p_{F_1} and $p_{F_1}(u_i)$ are on the same side of $\frac{c_r}{c_F}$.

Decision-theoretic SA with FORM (II)

The FORM EVPPI integral can be expressed as:

$$EVPPI_{F_1,i} = \left| c_F \Phi_2\left(-\beta, \bar{s}_i u_{i,t}, -s_i \alpha_i\right) - \Phi\left(\bar{s}_i u_{i,t}\right) \right|$$

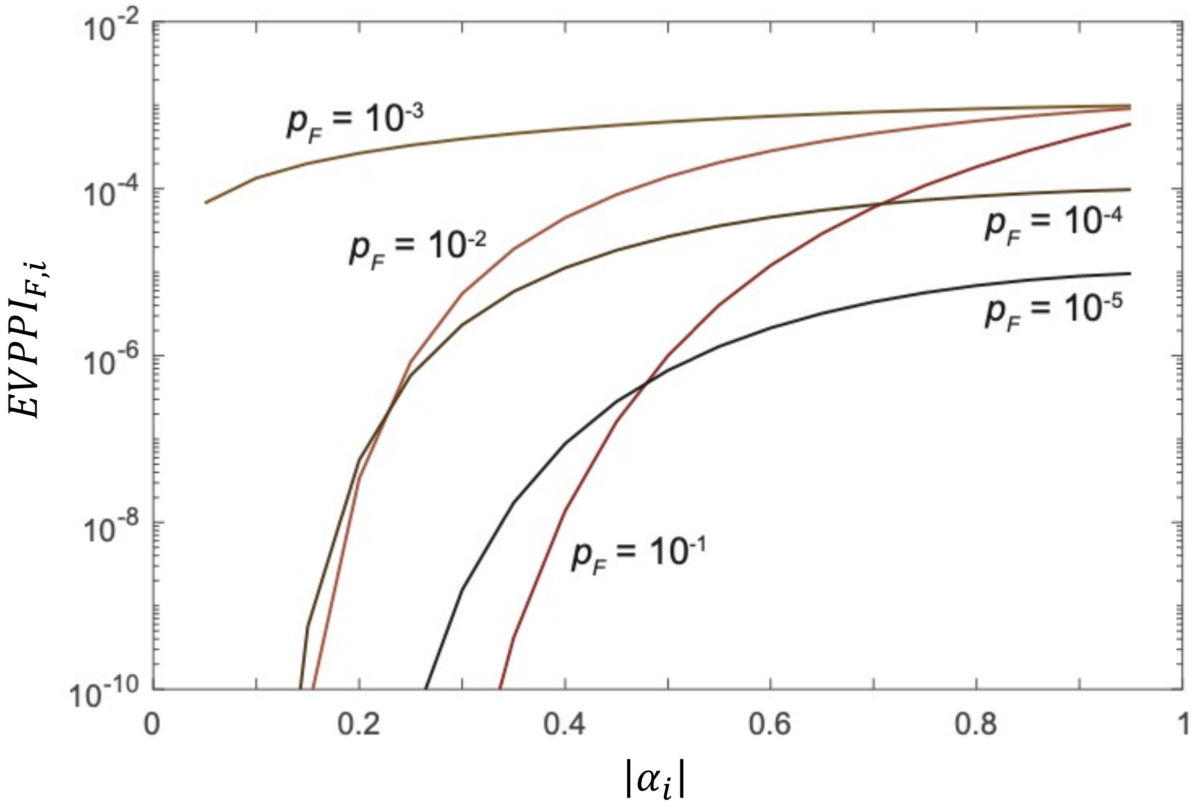
where $\Phi_2(\cdot,\cdot,r)$ is the bivariate standard normal CDF with correlation coefficient r,

$$u_{i,t} = \frac{1}{\alpha_i}\left[\sqrt{1-\alpha_i^2}\,\Phi^{-1}\left(\frac{c_r}{c_F}\right) + \beta\right]$$

$$\bar{s}_i = \operatorname{sgn}\left[\left(p_{F_1} - \frac{c_r}{c_F}\right)\alpha_i\right]$$

Illustration

FORM-based EVPPI vs. $|\alpha_i|$ for varying p_F values and $\frac{c_r}{c_F} = 10^{-3}$

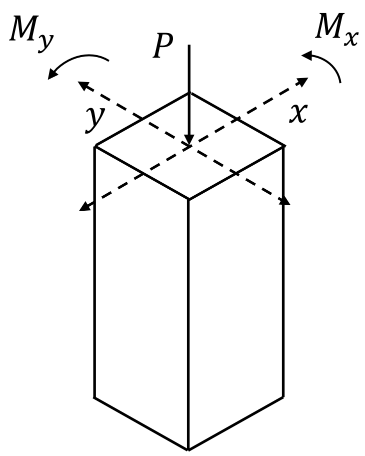


Observations

- The ranking of FORM α-factors is consistent with the one of the EVPPI of the FORM approximation
- The behavior of the EVPPI as a function of the FORM α-factors depends on the relation of p_F to c_r/c_F
- The absolute value of the EVPPI is highest when p_F is close to c_r/c_F

Example: Short column

Column subjected to biaxial bending and axial force.



Limit-state function:

$$g(\mathbf{x}) = 1 - \frac{M_x}{s_x Y} - \frac{M_y}{s_y Y} - \left(\frac{P}{AY}\right)^2$$

Deterministic parameters:

$s_x = 0.03\mathrm{m}^3$, $s_y = 0.015\mathrm{m}^3$, $A = 0.19\mathrm{m}^2$

Reference probability of failure:

$p_F = 4.85\times10^{-3}$

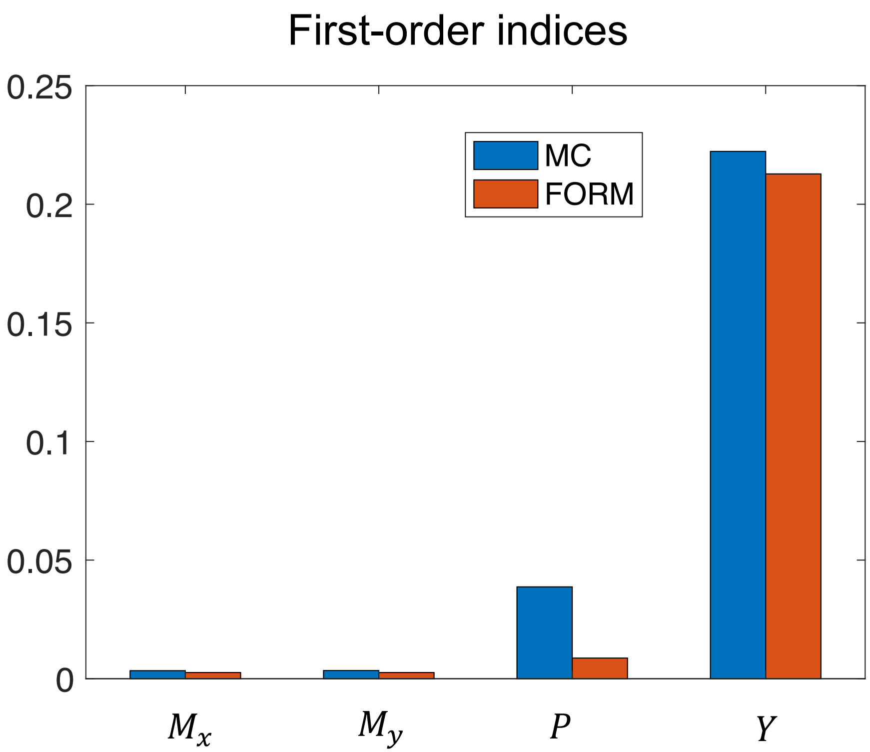
FORM estimate:

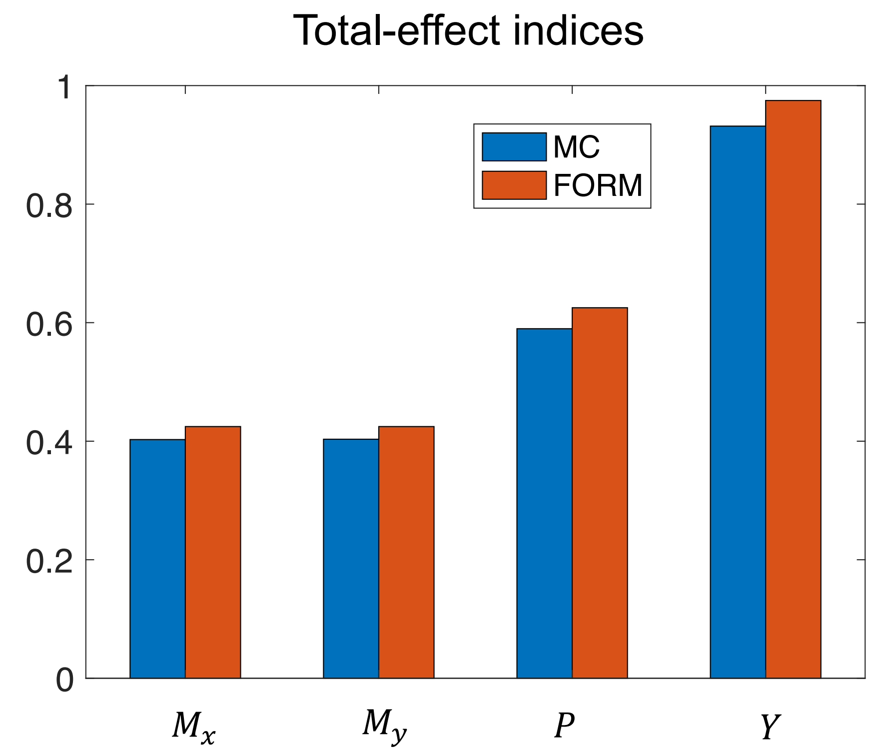
$p_F = 3.37\times10^{-3}$

Parameter	Distribution	Mean	CV
M_x [KNm]	Normal	250	0.3
M_y [KNm]	Normal	125	0.3
P [kN]	Gumbel	2500	0.2
Y [MPa]	Weibull	40	0.1

Example: Short column

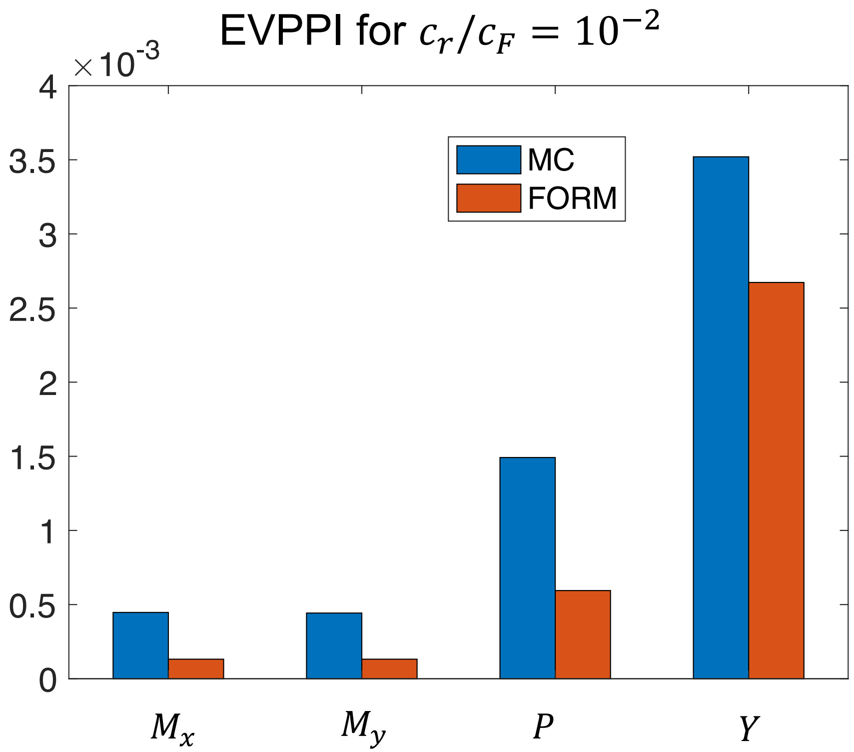
Variance-based reliability sensitivities

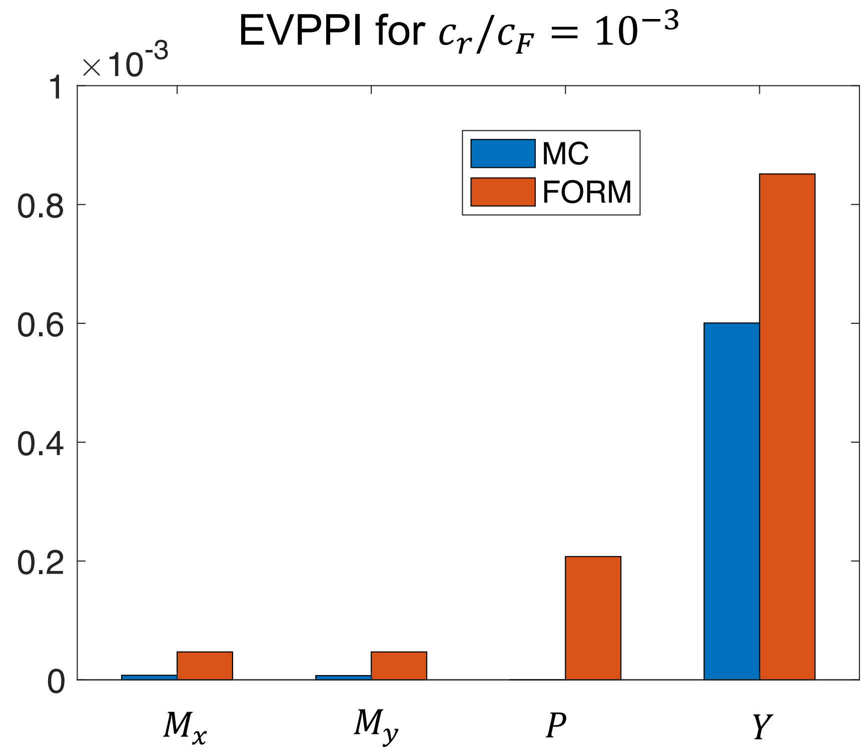




Example: Short column

Decision-theoretic reliability sensitivities





Conclusion

- Approximations of variance-based and decision-theoretic reliability sensitivities based on FORM
- The approximations depend only on the FORM reliability index and the α-factors
- The approximations of the sensitivities give consistent rankings with the absolute α-factors
- The derived expressions can be used for reliability sensitivities of mildly nonlinear component problems with independent inputs

Outlook

- Variance-based reliability analysis of dependent inputs and their relation the FORM reliability sensitivities of [Der Kiureghian 2005]
- The approximations can be extended to estimate reliability sensitivities of series- and parallel-system problems
- Investigation of other efficient strategies for variance-based and decision-theoretic reliability sensitivity analysis, e.g., based on surrogate modeling

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Variance-based reliability sensitivity analysis

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