Prolegomena to nonlinear discrete-time systems

DYNAMICS, CONTROL, and GEOMETRY

in honor of Bronislaw Jakubczyk

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thanks to Witold

thanks to Bronislaw

does the discrete-time control community still a minority !

Go back to the late seventies early eighties the "golden age" of nonlinear control theory (continuous time)

Pioneering work includes

H. J. Sussmann & V. Jurdjevic. *Controllability of nonlinear systems.* J. Diff. Equat. 1972.

H. J. Sussmann. *Orbits of families of vector fields and integrability of distributions*, Trans. AMS., 1973.

C. Lobry. *Controllability of nonlinear systems on compact manifolds*. SIAM J. Control, 1974.

R. W. Brockett. *Volterra series and geometric control theory.* Automatica, 1976.

R. Hermann & A. Krener. *Nonlinear controllability and observability.* IEEE Trans. on AC, 1977.

V. Jurdjevic & J.P. Quinn. *Controllability and stability.* Journal of Differential Equations, 1978.

C. Lesiak & A. J. Krener. *The existence and uniqueness of Volterra series for nonlinear systems.* IEEE Trans. on AC, 1978.

R. M. Hirschorn. *Invertibility of Nonlinear Control Systems.* SIAM J. Control, 1979.

B. Jakubczyk & W. Respondek. *On linearization of control systems.* Bull. Acad. Pol. Sci. Ser. Sci. Math., 1980

M.Fliess. *Fonctionnelles causales non lineaires et indetermines non commutatives.* Bulletin SMF1981.

A. Isidori, A. Krener, C. Gori-Giorgi, S. Monaco. *Nonlinear decoupling via feedback: A differential geometric approach.* IEEE Trans. on AC, 1981.

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A successful union of differential algebraic & differential geometric mathematical frameworks

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What about nonlinear discrete-time systems ????

⇒ Lack of well identified mathematical framework

⇒ Simplified cases are studied : bilinear & state-affine dynamics, quadratic nonlinearities ... in an essentially algebraic context

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From ideas emerged at the time of collaboration with Bronislaw :

- to revisit the impact of these tools in the nowadays knowledge of NLDT theory
- to project these ideas in the future of digital systems

Outline of the presentation

- Part 1 : the work with Bronislaw in the yersterday's context ...

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- Concluding comments : their impact in the nowadays digital environment !

All what I said founds its bases on the concerned literature and my own experience in straight collaboration with Salvatore Monaco, DIAG, Università Roma Sapienza with several talented PhD students !

Nonlinear discrete-time systems : typical difficulties

- Implicit representation over one step as a map :

$$x(k+1) = F(x(k), u(k)) = F_0(x(k)) + \sum_{i \geq 1} u^i(k) F_i(x(k))$$

$x \in \mathbb{R}^n$, $u \in U$ (nghb of 0 in $\mathbb{R}$), $F(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$ analytic.

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2. The drift term induces jumps
3. How to define local behaviours ?
4. What about some (semi)-group action ?

Beginning of the eighties : two leading questions

Given a DT system ($y \in \mathbb{R}^p$, $H(x) : \mathbb{R}^n \to \mathbb{R}^p$ analytic)

$$x(k+1) = F(x(k), u(k)); \qquad y(k) = H(x(k))$$

1. How to characterize the kernels of Volterra like input/state, input/output expansions ⇒ minimal realization problem?

$$\begin{aligned} y(k) =& H \circ F(\cdot, u(k-1)) \circ \cdots \circ F(x(0), u(0)) \\ =& w_0(k; x(0)) + \sum_{i>0} \sum_{\substack{\tau_1, \ldots, \tau_i = 0 \\ \tau_i \geq \tau_{i-1}}}^{k-1} w_i(k, \tau_1, \ldots, \tau_i; x(0)) u(\tau_1) \ldots u(\tau_i \end{aligned}$$

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2. How to characterize accessibility/reachability ⇒ controllability properties ?

How to compute the kernels of the Volterra expansions ?

$$
\begin{aligned}
y(k) =& H \circ F(\cdot, u(k-1)) \circ \cdots \circ F(x(0), u(0)) \\
=& w_0(k; x(0)) + \sum_{\tau_1=0}^{k-1} w_1(k, \tau_1; x(0()u(\tau_1) \\
&+ \sum_{\tau_1=0}^{k-1} w_2(k, \tau_1; x(0))u^2(\tau_1) + \sum_{\tau_2>\tau_1}^{k-1} w_2(k, \tau_1, \tau_2; x(0))u(\tau_1)u(\tau_2) + \ldots
\end{aligned}
$$

⇒ Necessity to distinguish $w_2(k, \tau_1; x(0))$ from $w_2(k, \tau_1, \tau_2; x(0))$

⇒ How to manage the nonlinearities in u ?

Algebraic and combinatorial methods (formal series expansions, generating series, rational series, ...) are rapidly shared.

E. Sontag. *Realization theory of discrete-time nonlinear systems. I-The bounded case.* IEEE TCS, 1979

M. Fliess. *Generating series for discrete-time nonlinear systems.* IEEE TAC, 1980

S. Monaco & DNC. *On the realization of nonlinear discrete-time systems.* SCL, 1984.

An algebraic existence result of realization

- Any discrete-time kernel can be realized by the cascade connection of finite dimensional state-affine systems :

$$x(k+1) = A_0 x(k) + \sum_{i\geq 1} u^i(k) A_i x(k); \qquad y(k) = Cx(k)$$

- Minimality corresponds to finite Hankel matrix rank or rationality of the associated generating series (*via the Hadamard Bochner Martin product*)

⇒ The DT version of CT results involve bilinear dynamics
⇒ Nonlinearity in u is unavoidable in discrete time !
⇒ Unified differential algebraic and combinatoric set up for both CT and DT

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How to characterize the set of accessible states ?

$$x(k+1) = F(x(k), u(k)) = x(k) + \bar{F}(x(k), u(k))$$
$$= F(\cdot, u(k)) \circ \cdots \circ F(x(0), u(0))$$

- as the image of the composition of functions over time steps ?
- as the orbit of certain (semi-)group actions ?

$\Rightarrow$ *Differential algebra and differential geometry are rapidly recognized as efficient mathematical frameworks*

Standard assumptions are rapidly shared

1. Invertibility of $F(\cdot, u)$ is required for any $u \in \mathcal{U}$ (or locally for $u = 0$) $\Rightarrow$ to go backward in time !

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2. There exists $\bar{u}$ such that $F(\cdot, \bar{u}) = Id \Rightarrow$ no drift term !
   $\Rightarrow$ A group action can be defined
   $\Rightarrow$ The orbit $\mathcal{O}(x(0))$ characterizes the weakly (forward and backward) accessible states from $x(0)$

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3. Infinitesimal transformations rely upon small variations of u (or around u) - derivation with respect to u replaces time derivative.
   $\Rightarrow$ Local behaviour associated to small variations of u makes sense

The CRAS note with Bronislaw (1984):

AUTOMATIQUE THÉORIQUE. — Orbites de pseudo-groupes de difféomorphismes et commandabilité des systèmes non linéaires en temps discret. Note de **Bronislaw Jakubczyk** et **Dorothée Normand-Cyrot,** présentée par Jacques-Louis Lions.

Remise le 23 janvier 1984.

On montre que l'orbite d'une famille de difféomorphismes locaux dépendant régulièrement d'un paramètre a une structure de variété dont on calcule le fibré tangent. On en déduit, pour les systèmes non linéaires en temps discret, un critère de rang de commandabilité faible, locale ou globale.

AUTOMATION (THEORETICAL). — Orbits of Pseudogroups of Local Diffeomorphisms and Controllability of Nonlinear Discrete-Time Systems.

We show that the orbit of a family of local diffeomorphisms depending smoothly on a parameter is a manifold, the tangent bundle of which is computed. This gives a rank-type criterion for (local and global) weak controllability of discrete-time systems.

Contemporary work includes:

B. Jakubczyk. *Invertibility realization of nonlinear discrete-time systems*. Princeton Conf. Info. Sci. Syst., 1980

M. Fliess & DNC. *A Lie theoretic approach to nonlinear discrete-time controllability via Ritt's formal differential groups*. SCL, 1981.

DNC. *Théorie et pratique des systèmes non linéaires en temps discret, Thèse d'Etat, 1983.*

Differential Ritt's group; *Ann of Math 1951*

Let Σ_d

$$x(k+1) = F(x(k), u(k)) = x(k) + \bar{F}(x(k), u(k))$$

invertible for any $u \in \mathcal{U}$ and assume $\bar{u}$ st $F(\cdot, \bar{u}) = Id$ (no drift term). Then:

- Σ_d is said weakly (forward and backward) controllable when any two states can be joined through a trajectory of the system (equivalently an element of the associated group)
- Σ_d is said (locally) weakly controllable (at $x(0)$) if the set of reachable states from $x(0)$ is a neighborhood of $x(0)$

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- Composition of functions defines a differential Ritt's group action $\star$ over $\{\bar{F}(\cdot, \bar{u})$ s.t. $\bar{u} \in \mathbb{R}^p\}$;

  $$F(\cdot, v) \circ F(\cdot, u) = (Id + \bar{F}(\cdot, v)) \circ (Id + \bar{F}(\cdot, u)) = Id + \bar{F}(\cdot, u) \star \bar{F}(\cdot, v)$$

  with the identity function as neutral element.

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- The orbit $\mathcal{O}(x(0))$ clearly characterizes the weakly (forward and backward) accessible states from $x(0)$.

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Controllability via small variations of u (or around u)

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$$\frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} F(x, u(0) + \epsilon)\bigg|_{F^{-1}(x,u(0))} := G(x, u(0))$$

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- and second controllability directions :

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} F(., u(1)+\epsilon) \circ F(x, u(0))\Big|_{F^{-1}(.,u(0))\circ F^{-1}(x,u(1))}$$
$$:= G(x, u(1))$$
$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} F(., u(1)) \circ F(x, u(0)+\epsilon)\Big|_{F^{-1}(.,u(0))\circ F^{-1}(x,u(1))}$$
$$:= Ad_{F(.,u(1))} G(x, u(0)) \quad (\textit{transported VF along } F(., u(1)))$$

- ...

A differential geometric apparatus is set in discrete time

Let :

$$G(\cdot,u) \qquad (\frac{\partial F(x,u)}{\partial u} = G(F(x,u),u)) \quad u \in \mathcal{U}$$

and its successive transports along any $F(x,v)$:

$$Ad_{F(\cdot,v)}G(\cdot,u) = \frac{\partial F(x,v)}{\partial x} \times G(.,u)\Big|_{F^{-1}(.,v)} \quad u,v \in \mathcal{U}$$

$$\Downarrow$$

$$\mathcal{L}\text{: } = \text{Lie}\{\tfrac{\partial^i}{\partial u^i}G(\cdot,u), \tfrac{\partial^{i+j}}{\partial v^i \partial u^j}Ad_{F(\cdot,v)}G(\cdot,u), \cdots\}$$

Result 1:

Notons $\mathcal{N}$ la famille de champs de vecteurs ($|_{u_0}$ est l'évaluation en u_0) :

$$\mathcal{N} = \left\{ g_u^j(x)|_{u_0},\ \frac{\partial^k}{\partial u^{i_k}\dots\partial u^{i_1}} g_u^j(x)|_{u_0},\ k \geqq 1,\ (i_k, \dots, i_2, i_1) \in \{1, \dots, m\} \right\}.$$

$\mathcal{L} = \mathrm{Lie}\, \mathcal{N}$ l'algèbre de Lie engendrée par $\mathcal{N}$ et $\mathcal{L}(x)$ l'espace vectoriel engendré par les $\{g(x);\, g \in \mathcal{N}\}$, $\mathcal{L}(x) \subset T_x M$.

Théorème. — *Si* $\dim \mathcal{L}(x_0) = \dim M$, *alors le système* (1) *est localement faiblement commandable en* x_0. *Si, pour tout* x *de* M, $\dim \mathcal{L}(x) = \dim M$, *alors le système* (1) *est faiblement commandable. Dans le cas analytique ces conditions sont aussi nécessaires.*

- *local at* $(x(0))$ *or global weak controllability is equivalent to* $\dim\mathcal{L}(x(0)) = n$ *(analytic case - multi input)*
- $\mathcal{N}$ *coincides with the family of* $\frac{\partial G(\cdot,u)}{\partial u^i}$ *previously defined when assuming the existence of* $\bar{u}$ *s.t.* $F(\cdot, \bar{u}) = Id$ *(drift term equal to the identity function)*
- *This assumption is relaxed when introducing the successive* $Ad_{F(\cdot,v)} G(\cdot, u)$

Result 2:

THÉORÈME 2. — *L'orbite $\mathscr{T}(x_0)$ a une structure unique de sous-variété immergée dans* M *telle que les applications ψ_σ : dom $\psi_\sigma \to \mathscr{T}(x_0)$ sont de classe C^k, la dimension de $\mathscr{T}(x_0)$ est égale au sup rang $d\psi_\sigma(\underline{u})$ (le maximum est pris sur toutes les suites finies σ et $\underline{u} \in$ dom ψ_σ). Dans le cas C^∞, $T_{x_0}(\mathscr{T}(x_0)) = \mathscr{M}(x_0)$. Dans le cas analytique, si* (H) *est vérifiée,* $T_{x_0}\mathscr{T}(x_0) = \mathscr{L}(x_0)$.

- *The orbit $\mathcal{O}(x(0))$ of the so defined pseudo group is a submanifold in $\mathcal{M}$.*
- *In the analytic case :*

$$dim T_{x(0)}\mathcal{O}(x(0)) = dim \mathcal{L}(x(0)) = n.$$

Further results in :

B. Jakubzyck & E. Sontag. *Controllability of nonlinear discrete-time systems: a Lie-algebraic approach*. SIAM, 1990.

F. Albertini & E. Sontag. *Discrete-Time Transitivity and Accessibility: Analytic Systems*, SIAM.1993

M. Barbero-Linan &B. Jakubczyk.*S*econd Order Conditions for Optimality and Local Controllability of Discrete-Time Systems . SIAM 2012

T. Mullari, Kotta, Z. Bartosiewicz, MA Sarafrazi, C. Moog, E. Pawluszewicz. *Weak reachability and controllability of discrete-time nonlinear systems: generic approach and singular points*, IJC 2018.

more · · ·

Part 2 of the presentation

What about further use of these VF ?

1. An alternative state space Differential/Difference representation
2. A new passivity notion in discrete time
3. The example of sampled dynamics

Part 2: first example

A new state space Differential/Difference representation allowing :

- An exponential form representation of the underlying group action;
- A geometric characterization of structural /control properties.

Differential/Difference Representation - DDR

$$x^+ = F_0(x), \quad x^+(0) := x^+$$
$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u)$$

- $F_0(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ models the drift evolution as a jump at time k;
- $G(\cdot, u) : \mathbb{R}^n \times U \to \mathbb{R}^n$ (a complete VF) models the rate of change of the state behaviour w.r.t. the control effect between k and $k+1$;

$$G(\cdot, u) = G_1(\cdot) + \sum_{i>1} \frac{u^i}{i!} G_{i+1}(\cdot)$$

- $x^+(\cdot) : \mathcal{U} \to \mathbb{R}^n$: a curve in $\mathbb{R}^n$ parametrized by u.

S. Monaco & DNC. *Discrete-time state representations, a new paradigm.* Springer, 1998.

Equivalence between DDR and representation as a map

Given

$$x^+ = F_0(x), \quad x^+(0) := x^+$$
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On the other side, invertibility of $F_0(\cdot)$ guarantees the existence of $G(\cdot, u)$

$L_{G(\cdot,v)}$: the Lie derivative associated with $G(\cdot, v)$

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Integration over $[0, u[$ from $x^+ = x^+(0)$ recovers $F(x, u)$:

$$\begin{aligned} x^+(u) = F(x,u) &= x^+(0) + \int_0^u G(x^+(v), v)\mathrm{d}v \\ &= F_0(x) + \int_0^u \frac{\partial F(x,v)}{\partial v}\mathrm{d}v \end{aligned}$$

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Integration over $[0, u[$ from $x^+ = x^+(0)$ recovers $F(x, u)$:

$$x^+(u) = F(x, u) = x^+(0) + \int_0^u G(x^+(v), v)\mathrm{d}v$$
$$= F_0(x) + \int_0^u \frac{\partial F(x, v)}{\partial v}\mathrm{d}v$$

Analogously :

$$y^+(u) = H(F(x, u)) = H(F_0(x)) + \int_0^u L_{G(\cdot,v)}H(x^+(v))\mathrm{d}v$$

On the other side, invertibility of $F_0(\cdot)$ guarantees the existence of $G(\cdot, u)$

$L_{G(\cdot,v)}$: the Lie derivative associated with $G(\cdot, v)$

Chronological exponential expansion of the flow

Given:

$$x^+ = F_0(x)$$
$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u) \qquad x^+(0) := x^+$$

integration $\downarrow$ *with respect to* u

$$x^+(u) = \phi_u^G(x^+(0)) = \overrightarrow{\exp} \int_0^u G(x^+(0), v)dv \qquad \textit{(chronological exponential)}$$

$$x^+(u) = x^+(0) + \sum_{m\geq 1} \int_0^u dv_1 ... \int_0^{v_{m-1}} dv_m G(., v_m) \circ ... \circ G(x^+(0), v_1)$$

$$\phi_.^G : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n \ \left(\frac{\mathrm{d}\phi_u^G(\cdot)}{\mathrm{d}u} = G(\phi_u^G(\cdot), u)\right) \ (\textit{the associated flow})$$

A. Agrachev & A. V. Gamkrelidze. *Chronological algebras and nonstationary vector fields.* Problemy Geometrii, 1980.

Computable exponential form of the flow

- ϕ_u^G :*(flow) admits an exponential representation:*

$$\phi_u^G(x) = e^{u\mathcal{G}(.,u)}(Id)\Big|_x$$

$$u\mathcal{G}(.,u) = \sum_{i\geq 1} \frac{u^i}{i!}\mathcal{B}_i = u\mathcal{B}_1 + \frac{u^2}{2}\mathcal{B}_2 + \frac{u^3}{3!}\mathcal{B}_3 + \frac{u^4}{4!}\mathcal{B}_4 + O(u^5)$$

- $\mathcal{B}_i$:*homogeneous Lie polynomial of degree i in the G_i*

$$\mathcal{B}_{i+1} = G_{i+1} + \sum_{0\leq k<i} \sum_{\substack{l_1,\ldots,l_j\geq 1 \\ \Sigma_p(l_p)+k=i}} \frac{(-1)^j i! b_j}{j!k!l_1!\ldots l_j!} ad_{\mathcal{B}_{l_1}} \circ \ldots \circ ad_{\mathcal{B}_{l_j}} G_{k+1}$$

$$\mathcal{B}_1(\cdot) = G_1(\cdot),\ \mathcal{B}_2(\cdot) = G_2(\cdot),\ \mathcal{B}_3(\cdot) = G_3(\cdot) + \frac{1}{2}[G_1(\cdot), G_2(\cdot)], \ldots$$

S. Monaco, DNC & C. Califano. *From chronological calculus to exponential representations of continuous and discrete-time dynamics: a Lie algebraic approach*, IEEE Trans. on AC, 1987

More about the Lie structure

Rewrite:

$$G(\cdot,u) = G^0(\cdot,u) = G_1^0 + \sum_{i\geq 1} \frac{u^i}{i!} G_i^0$$

$$G^p(\cdot,u) = G_1^p + \sum_{i\geq 1} \frac{u^i}{i!} G_i^p \quad \text{with} \quad G_i^p(\cdot) = Ad_{F_0^p} G_i^0(\cdot)$$

$$\phi_u^{G^p} = e^{u\mathcal{G}^p(\cdot,u)} = e^{\sum_{i\geq 1} \frac{v^i}{i!} \mathcal{B}_i(G_1^p(\cdot),\ldots,G_i^p(\cdot))}$$

Then :

$$F^{-1}(\cdot,u) = e^{-u\mathcal{G}^1(\cdot,u)}(\cdot)\Big|_{F_0^{-1}(x)}$$

$$Ad_{F(\cdot,v)} G^0(\cdot,u) = e^{-ad_{v\mathcal{G}^0(\cdot,v)}} G^1(\cdot,u) \in \mathcal{L}\{G_i^0, Ad_{F_0} G_i^0(\cdot), i \geq 1\}$$

$$\phi_v^{G^1} \circ \phi_u^{G^0} = e^{\mathcal{BCH}(v\mathcal{G}^1(\cdot,v),u\mathcal{G}^0(\cdot,0))}(Id)$$

$\mathcal{BCH}(X,Y) \in \mathcal{L}\{G_i^0(\cdot), Ad_{F_0} G_i^0(\cdot), i \geq 1\}$ *Baker-Campbell-Hausdorff exponent*

Accessibility from $x(0)$ in k steps

The set of forward accessible states from $x(0)$ in k steps is:

$$\mathcal{I}(k)(x(0)) = \{ e^{u(0)\mathcal{G}^{k-1}(\cdot,u(0))} \circ \cdots \circ e^{u(k-1)\mathcal{G}^{0}(\cdot,u(k-1))} \mathrm{Id}\big|_{F_0^k(x(0))} ; u(i) \in U \}.$$

Identically for the input/output response

$$\mathcal{Y}(k)(x(0)) = \{ e^{u(0)\mathcal{G}^{k-1}(\cdot,u(0))} \circ \cdots \circ e^{u(k-1)\mathcal{G}^{0}(\cdot,u(k-1))} \mathrm{H}\big|_{F_0^k(x(0))} ; u(i) \in U \}.$$

$\Rightarrow$The exponential form of the flow induces an explicit decomposition of each Volterra kernel in terms of the $Ad_{F_0^p}G_i$ and their Lie brackets;
$\Rightarrow$The group orbit ocharacterizes the weakly accessible states.

Revisiting the previous CRAS result :

Notons $\mathcal{N}$ la famille de champs de vecteurs ($|_{u_0}$ est l'évaluation en u_0) :

$$\mathcal{N} = \left\{ g_u^i(x)|_{u_0}, \frac{\partial^k}{\partial u^{i_k} \ldots \partial u^{i_1}} g_u^i(x)|_{u_0}, k \geq 1, (i_k, \ldots, i_2, i_1) \in \{1, \ldots, m\} \right\}.$$

$\mathcal{L} = \mathrm{Lie}\, \mathcal{N}$ l'algèbre de Lie engendrée par $\mathcal{N}$ et $\mathcal{L}(x)$ l'espace vectoriel engendré par les $\{g(x); g \in \mathcal{N}\}$, $\mathcal{L}(x) \subset T_x M$.

THÉORÈME. — *Si* $\dim \mathcal{L}(x_0) = \dim M$, *alors le système* (1) *est localement faiblement commandable en* x_0. *Si, pour tout* x *de* M, $\dim \mathcal{L}(x) = \dim M$, *alors le système* (1) *est faiblement commandable. Dans le cas analytique ces conditions sont aussi nécessaires.*

within the (F_0, G) formalism

$$\mathcal{L} = \mathrm{Lie}\{G_i, Ad_{F_0^p} G_i, \cdots; i \geq 1, p \in Z\}$$

The case of only one $G_1(x)$

$$x^+ = F_0(x)$$

$$\frac{\mathrm{d}(x^+(u))}{\mathrm{d}u} = G_1(x^+(u)) \qquad \text{with} \quad x^+ = x^+(0)$$

- $\phi_u^G(\cdot) = e^{uG_1}(\cdot) \qquad \Rightarrow \qquad F(x,u) = x^+(u) = e^{uG_1} Id\Big|_{F_0(x)}$
- *Additive one parameter Lie group action*

$$e^{uG_1(\cdot)} \circ e^{vG_1(\cdot)} = e^{(u+v)G_1(\cdot)} \quad \text{with,} \qquad e^{uG_1(\cdot)} \circ e^{-uG_1(\cdot)} = \mathrm{I}$$

- $\mathcal{L} = \mathrm{Lie}\{G_1(\cdot), Ad_{F_0^p(\cdot)} G_1(\cdot), \ldots ; p \in Z\}$ *controllability Lie algebra*

S. Monaco & DNC. *Geometric properties of a class of nonlinear discrete-time dynamics.* ECC, 1997.

First conclusions

The DDR is efficient :

- to set a differential geometrical framework;
- to describe the dynamics over several steps;
- to study and formalize structural properties (accessibility, invariance, normal forms, etc) & control properties (feedback linearization, decoupling, Lyapunov design, etc)
- to characterize physical properties (e.g., passivity)
- $\cdots$

Further the DDR provides

- an alternative way for discrete-time modeling
- a suitable representation to deal with hybrid or switched dynamics

S. Monaco & DNC. *Invariant distributions for nonlinear discrete time systems.* SCL, 1984.

B. Jakubczyk. *Feedback linearization of discrete-time systems.* SCL, 1987.

C. Califano, S. Monaco & DNC. *On the problem of feedback linearization.* SCL, 1999.

Part 2: second example

What about passivity in discrete-time ?

What about passivity in discrete-time ?

A new passivity concept for discrete-time systems !

Passivity in discrete time : a natural definition

The system

$$x(k+1) = F(x(k), u(k))$$
$$y(k) = H(x(k))$$

is *passive* with storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ if $\forall u(k) \in \mathcal{U}$

$$V(x(k+1)) - V(x(k)) \leq y^\top(k)u(k)$$

What about the case of LTI systems ?

C. I. Byrnes & W. Lin. *Discrete-time lossless systems, feedback equivalence and passivity*. CDC, 1993.

Usual passivity in DT requires a direct I/O link

$$\begin{aligned} x(k+1) =& Ax(k) + Bu(k) \\ y(k) =& Cx(k) + (D^\top + D)u(k) \end{aligned}$$

is passive with $V(x) = \frac{1}{2}x^\top Px$ with $P \succ 0$ iff

$$x^\top A^\top PAx - x^\top Px + 2x^\top(A^\top PB - C^\top)u + u^\top(B^\top PB - D^\top - D)u \leq 0$$

$$\downarrow\uparrow$$

$$\left[\begin{array}{cc} A^\top PA - P & A^\top PB \\ B^\top PA & B^\top PB - D - D^\top \end{array}\right] \preceq 0$$

$$\implies D \neq 0$$

A new passivity notion in discrete time

$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u); \qquad x^+ = F_0(x); \qquad y(k) = H(x(k))$$

The system is u-average passive if $\exists V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $V(0) = 0$ (*storage function*) s.t. $\forall u(k) \in \mathcal{U}$

$$\begin{aligned} V(x(k+1) - V(x(k)) =& V(F_0(x(k))) - V(x(k)) + \int_0^{u(k)} L_{G(\cdot,v)} V(x^+(v))\mathrm{d}v \\ &\leq \int_0^{u(k)} H(x^+(v))\mathrm{d}v) := u(k) H_{\mathrm{av}}(x(k), u(k)) \end{aligned}$$

$$H_{\mathrm{av}}(x,u) = \frac{1}{u}\int_0^u H(x^+(v))\mathrm{d}v. \qquad (u - \textit{average} \quad \textit{output})$$

S. Monaco & DNC. *Nonlinear average passivity and stabilizing controllers in discrete time.* SCL, 2011.

Equivalence with usual passivity

u-average passivity of

$$x^+ = F_0(x)$$
$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u); \qquad x^+(0) = x^+$$
$$y = H(x)$$

is equivalent to usual passivity of

$$x^+ = F_0(x)$$
$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u); \qquad x^+(0) = x^+$$
$$y = H_{av}(x, u)$$

$H_{av}(\cdot, u)$ *introduces a direct input-output link by construction*

Stability and u-Average Passivity

$$x^+ = F_0(x)$$
$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u); \qquad x^+(0) = x^+$$

- Assuming the dynamics Lyapunov stable with positive Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ then it is u-average passive with storage function $V(\cdot)$, with respect to the new output :

$$H(x, u) = \mathrm{L}_{G(\cdot,u)} V(x)$$

$$V(F(x,u)) - V(x) = V(F_0(x)) - V(x) + \int_0^u \mathrm{L}_{G(\cdot,v)} V(x^+(v))\mathrm{d}v \leq \int_0^u \mathrm{L}_{G(\cdot,v)} V(x^+(v))\mathrm{d}v$$

S. Monaco, DNC & F. Tiefensee. *Sampled-data stabilizing feedback: a PBC approach*. IEEE TAC 2011

Stability and u-Average Passivity

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$$V(F(x,u)) - V(x) = V(F_0(x)) - V(x) + \int_0^u \mathrm{L}_{G(\cdot,v)} V(x^+(v))\mathrm{d}v \leq \int_0^u \mathrm{L}_{G(\cdot,v)} V(x^+(v))\mathrm{d}v$$

- Any feedback $u = u(x)$ solution to the damping equality

$$u + K H_{\mathrm{av}}(x, u) = 0 \quad , K > 0$$

ensures asymptotic stability in closed loop provided ZSD holds.

S. Monaco, DNC & F. Tiefensee. *Sampled-data stabilizing feedback: a PBC approach.* IEEE TAC 2011

Second conclusions

- u-average passivity relaxes the necessity of an input-output link;
- construction of a passivating output $y = H(x, u) = \mathrm{L}_{G(\cdot,u)}V(x)$ is made possible;
- the resulting u-average passivity based stabilizing feedback is of the $\mathrm{L}_{G(\cdot,u)}V$ type;
- any passivity based design can be revisited (damping, passivation and stabilization of cascade (backstepping/feedforwarding), energy shaping, Hamiltonian structures) for discrete-time dynamics.

M. Mattioni, S.Monaco & DNC.*Lyapunov stabilization of discrete-time feedforward dynamics*.IEEE-CDC. 2017

The case of sampled-data systems !

How to conciliate continuous-time and discrete-time addicts opening wide perspectives to digital design !

Sampled-data systems : the associated flow

Let

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)) \quad \text{assume} \qquad u(t) = u(k); t \in [k\delta, (k+1)\delta[$$

- Associated flow:
  $x(k+1) = e^{\delta(f+u(k)g)}x(k) = F^{\delta}(x(k), u(k)) = \Phi^{G^{\delta}}_{u(k)}(F^{\delta}_0(x(k))$
- Exponential form : $\Phi^{G^{\delta}}_{u}(\cdot) = e^{u\mathcal{G}^{\delta}(.,u)}(\cdot)$
- Lie series exponent: $u\mathcal{G}^{\delta}(.,u) = \sum_{i\geq 1} \frac{u^i}{i!} B_i(G^{\delta}_1, \ldots, G^{\delta}_i)$

$$u\mathcal{G}^{\delta}(.,u) = G^{\delta}_1 + \frac{u^2}{2!}G^{\delta}_2 + \frac{u^3}{3!}(G^{\delta}_3 + 1/2\text{ad}_{G^{\delta}_1}G^{\delta}_2) + \frac{u^4}{4!}(G^{\delta}_4 + \text{ad}_{G^{\delta}_1}G^{\delta}_3) + \ldots$$

Sampled-data systems : the associated DDR

The DDR is described as

$$x^+ = F_0^\delta(x)$$
$$\frac{\mathrm{d}(x^+(u))}{\mathrm{d}u} = G^\delta(x^+(u), u) = G_1^\delta(x) + \sum_{i>0} \frac{u^i}{i!} G_i^\delta(x)$$

with

$$F_0^\delta(x) = e^{\delta \mathrm{L}_f}(x)$$
$$G^\delta(x,u) = \int_0^\delta e^{-s\ ad_{f+ug}} g(x)\mathrm{d}s = \delta g(x) + \sum_{i>0} \frac{(-1)^i \delta^{i+1}}{(i+1)!} ad^i_{f+ug} g(x)$$

Iterative computation of the $G_i^\delta(\cdot)$

Integro-differential formula :

$$\frac{\partial G^\delta(\cdot,u)}{\partial u} = \int_0^\delta [\frac{\partial G^s(\cdot,u)}{\partial s}, G^s(\cdot,u)]\mathrm{d}s$$

$$\Downarrow$$

$$G_1^\delta(\cdot) = \int_0^\delta e^{-s\ ad_f} g(\cdot)\mathrm{d}s \qquad \in \mathcal{L}_0 \qquad \textit{Lie ideal of } Lie\{f,g\}$$

$$G_2^\delta(\cdot) = \int_0^\delta [\frac{\partial G_1^s(\cdot)}{\partial s}, G_1^s(\cdot)]\mathrm{d}s = -G_1^\delta(\cdot) * G_1^\delta(\cdot) \in \mathcal{L}_0^2 := [\mathcal{L}_0, \mathcal{L}_0]$$

$*$ *chronological product*

$$G_i^\delta(\cdot) \in \mathcal{L}_0^i \quad i \geq 1$$

$$Ad_{F_0^\delta(\cdot)} G_i^\delta(\cdot) = e^{-\delta ad_f} G_i^\delta(\cdot) \in \mathcal{L}_0^i$$

S. Monaco & DNC. *A unifying representation for nonlinear discrete-time and sampled dynamics.* Journal of Mathematical Systems, Estimation and Control, 1997.

Third conclusions

In the sampled-data context, the DDR allows:

- to set up a systematic framework to carry out constructive methods for both analysis and control purposes;
- to address the problem of studying the effect of sampling (preservation Yes/No of continuous-time properties);

"The sampled-data equivalent to an passive input-affine continuous-time system is average passive !!!!"

The DDR goes beyond discrete time as a source to handle :

- continuous time-varying dynamics ;
- generally nonlinear continuous-time dynamics.

Finally !

Bon Anniversaire Bronislaw

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- A unifying framework to bridge the gap among continuous and discrete-time systems through sampled data dynamics does exist !

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- Many continuous-time experts do share such idea !

Finally !

Bon Anniversaire Bronislaw

- A unifying framework to bridge the gap among continuous and discrete-time systems through sampled data dynamics does exist !
- These methodologies take place into the renewed context of digital systems at large !
- Many continuous-time experts do share such idea !
- The best has yet to come !