Local Stabilization of Nonlinear Systems through the Reduction Model Approach

Frederic Mazenc Michael Malisoff

***Abstract*—We study a general class of nonlinear systems with input delays of arbitrary size. We adapt the reduction model approach to prove local asymptotic stability of the closed loop input delayed systems, using feedbacks that may be nonlinear. Our Lyapunov-Krasovskii functionals make it possible to determine estimates of the basins of attraction for the closed loop systems**

***Key Words*— delay, nonlinear, reduction model, stabilization**

I. Introduction

The reduction model approach is a well-known stabilization technique for systems with input delays. It originated in [1] and has been studied in many works, e.g., [2], [3], [4], [5], and [6]. It is effective for stabilizing linear time-invariant systems with arbitrarily long pointwise or distributed input delays. However, the approach does not directly apply to nonlinear systems; it is extended by introducing an extra dynamic (which gives the 'state predictor') whose initial condition is given by an implicit equation (as is done in [7], [8], [9], and [6, Chapt. 6, p. 128]), and only a few recent works adapt it to time varying systems [10]. This is a limitation, because many systems are nonlinear and lead to the stabilization of time varying nonlinear systems when a trajectory has to be tracked. Moreover, the work [11] is limited to globally Lipschitz nonlinear systems, and it has a restriction on the size of the delays. See also [12] and [13] for stabilization of nonlinear systems with arbitrarily long input delays when the systems have special structures, and [14] for compensation of arbitrarily long input delays under input sampling based on prediction.

These remarks motivate our work. We show that the reduction model approach can be used to locally asymptotically stabilize a large family of nonlinear time varying systems of the form $\dot{x}(t) = A(t)x(t) + B(t)u(t-\tau) + F(t, x(t))$, with arbitrarily long constant known input delays τ, where F is of order 2 in x at the origin. Our key assumption is the stabilizability of a linear approximation of the closed loop system at 0. Under this assumption, the result seems intuitively obvious. However, to the best of the authors' knowledge, it has never been rigorously established. In particular, the stability of the closed loop system we obtain cannot be proven by applying the Hartman-Grobman theorem, which only applies to ordinary differential equations; see [15, Chapt. 1]. One of the crucial benefits offered by our result is that it yields asymptotically stable closed loop systems for which one can determine a suitable subset of the basin of attraction of the closed loop systems. This information is valuable, because it gives a guarantee that some solutions converge to the origin. We estimate the basin of attraction by building a Lyapunov-Krasovskii functional. It is different from the one in [16], but can be combined with it to establish ISS results. See also [17] for estimates of the basins of attraction for time invariant nonlinear systems with predictor feedbacks, under an ISS assumption on the closed loop systems with undelayed controllers. The predictor feedbacks in [17] can be implemented using numerical methods but are totally different from ours, so our work can be viewed as complementary to [17]. Our work is mainly a methodological development, rather than a specific real-world application or experiments. However, input delays naturally arise from measurement and transport phenomena, and our assumptions are very general, so we anticipate that our work can have considerable benefits when applied to mechanical systems where latencies commonly occur.

The rest of this note is organized as follows. We give our definitions in Section II. In Section III, we show how the class of systems we study naturally arises in tracking problems. We state our main result in Section IV, and we prove it in Section V. In Section VI, we discuss a large class of examples where the estimates of the basins of attraction become arbitrarily large when the input delays converge to zero. In Section VII, we illustrate our result in a worked out example. We conclude in Section VIII with a summary of our findings.

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F. Mazenc is with EPI DISCO INRIA-Saclay, L2S, CNRS-Supelec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France. Frederic.Mazenc@lss.supelec.fr.

M. Malisoff is with the Department of Mathematics, 303 Lockett Hall, Louisiana State University, Baton Rouge, LA 70803-4918, USA. malisoff@lsu.edu. Malisoff was supported by NSF-ECCS Grant 1102348.

II. Definitions and Notation

We let $n \in \mathbb{N}$ be arbitrary and I_n denote the identity matrix in $\mathbb{R}^{n\times n}$, and $|\cdot|$ be the usual Euclidean norm of matrices and vectors. For square matrices M_1 and M_2 of the same size, we write $M_1 \geq M_2$ to mean that $M_1 - M_2$ is nonnegative definite. For each integer $r \geq 1$, let C^r denote the set of all functions whose partial derivatives up through order r exist and are continuous, and C^0 denotes the set of all continuous functions, when the domains and ranges are clear from the context. When we want to emphasize the domains and ranges, we use $C^r(\mathcal{U}, \mathcal{V})$ to denote the set of all C^r functions having domain $\mathcal{U}$ and range $\mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are suitable subsets of Euclidean spaces. For any constant $\tau \geq 0$ and any continuous function $\varphi : [-\tau, \infty) \to \mathbb{R}^n$ and all $t \geq 0$, we define the function φ_t by $\varphi_t(\theta) = \varphi(t+\theta)$ for all $\theta \in [-\tau, 0]$, i.e., the translation operator. Let $\mathcal{K}_\infty$ be the set of all C^0 functions $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and γ is strictly increasing and unbounded. Given subsets S_1 and S_2 of Euclidean spaces, we say that a function $J : S_1 \times S_2 \to \mathbb{R}^p$ is locally Lipschitz with respect to its second argument provided for each compact set $E \subseteq S_2$, there is a constant L_E such that $|J(p,x) - J(p,y)| \leq L_E|x-y|$ for all $p \in S_1$ and all $x \in E$ and $y \in E$. We say that J is strictly increasing in its second argument provided the function $Y(x) = J(p,x)$ is strictly increasing for each $p \in S_1$; we define strictly increasing and nondecreasing in either argument in a similar way. We say that J has order 2 in y at the origin provided there is a continuous function α such that $|J(p,y)| \leq |y|^2\alpha(|y|)$ for all $(p,y) \in S_1 \times S_2$. We sometimes omit arguments of functions when the arguments are clear from the context.

III. Motivation: Tracking Problem

In this section, we explain how the problem of tracking a trajectory may lead to the problem we solve in the next section. Consider a time varying nonlinear system

$$\dot{\xi}(t) = g(t, \xi(t)) + B(t)\mu(t-\tau)\ , \tag{1}$$

where the state ξ is valued in $\mathbb{R}^n$, the control μ is valued in $\mathbb{R}^p$, $\tau \geq 0$ is a known constant delay, $g = (g_1, g_2, \ldots, g_n)^\top$ is a nonlinear function of class C^2, and B is a continuous function. The dimensions n and p are arbitrary. We assume that (1) is forward complete for all measurable locally essentially bounded choices for μ, so $\xi(t)$ is defined for all nonnegative times for all such μ's. We also assume that there is a nondecreasing function γ such that

$$\begin{array}{l}\max\left\{\left|\frac{\partial^2}{\partial\xi^2}g_i(t,\xi)\right| : |\xi| \leq q, t \geq 0, i \in \{1,2,\ldots,n\}\right\}\\ \leq\ \gamma(q)\end{array} \tag{2}$$

for all $q \geq 0$, which exists when g is C^2 and periodic in t.

The objective is to follow an admissible trajectory ξ_r of class C^1, meaning the dynamics for $x = \xi - \xi_r$ should be asymptotically stable. By admissible, we mean that there is a known continuous function $\mu_r(t)$ such that $\dot{\xi}_r(t) = g(t, \xi_r(t)) + B(t)\mu_r(t)$ for all $t \geq 0$. In particular, this means that $\xi_r(t)$ is defined for all $t \geq 0$. We assume that ξ_r is a known bounded function.

Let $x(t) = \xi(t) - \xi_r(t)$ and $\mu(t-\tau) = u(t-\tau) + \mu_r(t)$. Then the error equation is

$$\dot{x}(t) = G(t, x(t)) + B(t)u(t-\tau) \ , \tag{3}$$

where $G(t,x) = g(t, x+\xi_r(t)) - g(t, \xi_r(t))$. Notice that $G(t,x) = \int_0^1 \frac{\partial g}{\partial x}(t, \ell x + \xi_r(t))x\mathrm{d}\ell$, so $G(t,x) = \frac{\partial g}{\partial x}(t, \xi_r(t))x + F(t,x)$, where

$$F(t,x) = \int_0^1 \left(\frac{\partial g}{\partial x}(t, \ell x + \xi_r(t)) - \frac{\partial g}{\partial x}(t, \xi_r(t))\right) x \,\mathrm{d}\ell \tag{4}$$

holds for all t and x.

Applying the Mean Value Theorem and using (2) and the boundedness of ξ_r, we can find a function $\alpha \in C^0$ such that $|F(t,x)| \leq |x|^2\alpha(|x|)$. Since ξ_r can depend on t, the system (3) is time varying, even if g is time-invariant and B is constant. This motivates the study of systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t-\tau) + F(t, x(t)), \tag{5}$$

where F is of order 2 in x at the origin, which will be our focus for the rest of this note.

IV. Statement of Main Result

We state our main result for (5), where x is valued in $\mathbb{R}^n$, the control u is valued in $\mathbb{R}^p$ and is to be specified, $\tau \geq 0$ is a given constant delay, and F is a nonlinear function. The dimensions n and p are arbitrary. The functions A, B and F are continuous, and F is locally Lipschitz with respect to x. The set of all initial conditions we consider is $E_0 = \left\{(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)\right\}$, so the initial times for our trajectories are always 0. Let $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ be the fundamental solution associated with A. Then $\lambda(t_0, t_0) = I_n$ and $\frac{\partial \lambda}{\partial t}(t, t_0) = A(t)\lambda(t, t_0)$ hold for all real numbers t and t_0. We introduce the following assumptions:

Assumption 1. *(i) There is a continuous, positive valued, nondecreasing function h such that*

$$|\lambda(t, l)B(l)| \leq h(\tau) \ \text{ for all } t \in \mathbb{R} \text{ and } l \in [t, t+\tau] \ . \tag{6}$$

(ii) There is a constant $a^+ \geq 0$ such that $\sup_{t\in\mathbb{R}} |A(t)| \leq a^+$. □

Assumption 1 always holds when B is bounded and A is constant, so for instance, it holds for the one-dimensional system

$$\dot{x}(t) = x(t) + u(t-\tau) + lx^2(t)\sin(x(t)) \tag{7}$$

where $u \in \mathbb{R}$ is the input, τ is a positive constant delay, and l is a positive constant. In the case of (7), we can take $A = 1$, $B = 1$, $\lambda(t, t_0) = e^{t-t_0}$, and $F(t,x) = lx^2\sin(x)$, so Assumption 1 holds with $h(\tau) = 1$. To ease the readability of our technical assumptions, we will explain how the example (7) satisfies our assumptions, after we introduce each of our three assumptions. Our next assumption is:

Assumption 2. *There are a continuous function $K : [0,\infty)^2 \to \mathbb{R}^{p\times n}$, a nondecreasing continuous function $k : [0,\infty) \to (0,\infty)$, an everywhere positive definite and symmetric function $Q : [0,\infty)^2 \to \mathbb{R}^{n\times n}$ of class C^1 with respect to its first argument, and continuous functions $q_i : [0,\infty) \to (0,\infty)$ for $i = 1,2,3$ such that $|K(t,\tau)| \leq k(\tau)$ for all $(t,\tau) \in [0,\infty)^2$, and such that with the choices $H(t,\tau) = A(t) + \lambda(t, t+\tau)B(t+\tau)K(t,\tau)$ and $R(t,\tau,s) = s^\top Q(t,\tau)s$, the following two conditions are satisfied for all $\tau \geq 0$: (i) Along all trajectories of $\dot{s}(t) = H(t,\tau)s(t)$, we have $\dot{R}(t,\tau,s(t)) \leq -q_1(\tau)R(t,\tau,s(t))$ and (ii) the bounds*

$$q_2(\tau)I_n \leq Q(t,\tau) \quad \textit{and} \quad |Q(t,\tau)| \leq q_3(\tau) \tag{8}$$

are satisfied for all $t \geq 0$. □

Assumption 2 holds for (7) as well. In fact, by choosing $K(t,\tau) = -2e^\tau$, we obtain $H(t,\tau) = 1 - e^{-\tau}2e^\tau = -1$, so Assumption 2 is satisfied with $Q(t,\tau) = \frac{1}{2}$, $q_1(\tau) = 2$, $q_2(\tau) = q_3(\tau) = \frac{1}{2}$, and $k(\tau) = 2e^\tau$. Finally, we assume:

Assumption 3. *There are two continuous functions f_1 and f_2 that are locally Lipschitz with respect to their last argument, and continuous functions α_1 and α_2, such that*

$$F(t,x) \ = \ \lambda(t, t+\tau)B(t+\tau)f_1(t,\tau,x) + f_2(t,x) \ \text{ and } \tag{9}$$

$$\begin{aligned} |f_1(t,\tau,x)| \ &\leq \ |x|^2\alpha_1(\tau, |x|^2) \ \text{ and } \\ |f_2(t,x)| \ &\leq \ |x|^2\alpha_2(|x|^2) \end{aligned} \tag{10}$$

for all $t \in \mathbb{R}$, $\tau \geq 0$, and $x \in \mathbb{R}^n$. Also, $\beta_3(\tau, m) = m\alpha_1(\tau, m^2)$ is strictly increasing and unbounded in m, and $\beta_4(m) = m\alpha_2(m^2)$ is nondecreasing in m. Finally, there are continuous functions $\theta_1 : [0,\infty)^2 \to (0,\infty)$ and $\theta_2 : [0,\infty) \to (0,\infty)$ such that

$$|\alpha_1(\tau, b+c) - \alpha_1(\tau, c)| \leq b\theta_1(\tau, b+c) \tag{11}$$

$$|\alpha_2(b+c) - \alpha_2(c)| \leq b\theta_2(b+c) \tag{12}$$

are satisfied for all $b \geq 0$ and $c \geq 0$. □

To see why (7) satisfies Assumption 3, note that for (7), the fact that $\lambda(t, t+\tau)B$ is invertible implies that one can choose $f_2 = 0$ and $f_1(t,\tau,x) = le^\tau x^2\sin(x)$. Then we can satisfy Assumption 3 for (7) by taking $\alpha_1(\tau, m) = le^\tau$ and $\alpha_2(m) = 0$ for all m and τ.

Returning to the general system (5), it follows from Assumptions 2-3 that for any constant $\tau > 0$ and

$$\alpha_3(\tau, m) = \frac{q_3(\tau)}{\sqrt{q_2(\tau)}}\alpha_2(m) + 2a\alpha_1(\tau, m) \ , \tag{13}$$

where a is any constant such that

$$0 < a \leq \frac{q_1(\tau)\sqrt{q_2(\tau)}}{8k(\tau)} \ , \tag{14}$$

there are unique positive values $v_1(\tau)$ and $v_2(\tau)$ (which also depend on a) such that

$$v_1(\tau)\alpha_3\left(\tau, \frac{4}{q_2(\tau)}v_1^2(\tau)\right) = \frac{q_1(\tau)q_2(\tau)}{16} \ \text{ and } \tag{15}$$

$$v_2(\tau)\alpha_3\left(\tau, \frac{4h^2(\tau)}{a^2}v_2^2(\tau)\right) = \frac{a^2}{4\tau h^2(\tau)} \ . \tag{16}$$

The existence of unique values $v_1(\tau)$ and $v_2(\tau)$ follows because $\beta_3(\tau, m)$ is strictly increasing and unbounded in m and $\beta_4(m)$ is nondecreasing, so $m\alpha_3(\tau, m^2)$ is strictly increasing and unbounded in m. The choice of α_3 in (13) will become clear when we prove:

Theorem 1. *Let $\tau > 0$ be any constant and Assumptions 1-3 hold. Let a be any constant satisfying (14), and set $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ where v_1 and v_2 are as above. Then, for each initial function $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$ satisfying*

$$\begin{aligned} &\sqrt{q_3(\tau)}\left|\phi_x(0) + \int_{-\tau}^0 \lambda(0, m+\tau)B(m+\tau)\phi_u(m)\mathrm{d}m\right| \\ &+ \tfrac{a}{\tau}\int_{-\tau}^0 (m+2\tau)|\phi_u(m)|\mathrm{d}m < v(\tau), \end{aligned} \tag{17}$$

the unique solution of (5), in closed loop with

$$\begin{aligned} u(t) = &-f_1(t,\tau,x(t)) + K(t,\tau)\big[x(t) + \\ &\int_{t-\tau}^t \lambda(t, m+\tau)B(m+\tau)u(m)\mathrm{d}m\Big] , \end{aligned} \tag{18}$$

converges to 0 as $t \to \infty$. Moreover, (18) locally asymptotically stabilizes (5) to 0. □

Remark 1. *We comment that our control (18) agrees with the standard predictor controller in the linear time invariant case where $f_1 = f_2 = 0$ and A and B are constant. The extra term $-f_1(t,\tau,x(t))$ is used to compensate part of the nonlinearity of the system (5). Assumption 2 is a generalization of the standard assumption that (A,B) is a stabilizable pair, which is the special case of Assumption 2 where $\tau = 0$, A and B are constant, and where K and Q can also be taken to be constant. However, we allow the delay $\tau > 0$ to be as large as we want. On the other hand, since the q_i's are continuous positive valued functions of the delay, they have positive upper and lower bounds over all $\tau \in [0, \tau_M]$ for any constant τ_M. Also, the function k from Assumption 2 is nondecreasing in τ. Hence, if we are only concerned with a bounded set $[0, \tau_M]$ of possible values for τ, then we can assume in Assumption 2 that the q_i's and k are all positive constants, by replacing them by the constants $\min\{q_1(\tau) : 0 \le \tau \le \tau_M\}$, $\min\{q_2(\tau) : 0 \le \tau \le \tau_M\}$, $\max\{q_3(\tau) : 0 \le \tau \le \tau_M\}$, and $k(\tau_M)$ without relabeling. These observations will be key to our proof in Section VI that for important special cases, our estimate of the domain of attraction becomes arbitrarily large when $\tau \to 0^+$.* □

Remark 2. *Assumptions 1-2 always hold when A and B are constant provided (A,B) is stabilizable. Indeed, in that case $\lambda(t,t_0) = e^{(t-t_0)A}$, so the stabilizability of (A,B) is equivalent to the stabilizability of $(A, \lambda(t,t+\tau)B)$. Also, when the α_i's are C^1, the existence of functions θ_i satisfying the requirements from Assumption 3 follows from the Mean Value Theorem, since Assumption 2 only requires (11)-(12) for nonnegative b's and c's. Since F is of order 2 in x at 0, we can always satisfy Assumption 3 with $f_1 = 0$ and $f_2 = F$. However, these choices may lead to a conservative estimate of the size of the basin of attraction; see the example in Section VII. Our use of a feedback control with distributed terms is motivated by the facts that τ is arbitrary and $\dot{\xi}(t) = A(t)\xi(t)$ may be exponentially unstable. In general, the explicit expression for λ is unknown, but it can be computed in many important cases. This is the case in particular if A is constant or $n = 1$. We illustrate Theorem 1 in Section VII.* □

Remark 3. *In conjunction with our local asymptotic stability result, we have boundedness of the control from Theorem 1, along all of the closed loop trajectories.* □

V. Proof of Theorem 1

Throughout the proof, we consider any solution of (5) in closed loop with (18) for any initial condition satisfying the requirements (17) of Theorem 1, and any constant delay $\tau \ge 0$.

First part: new representation of the system. Let t_e be any positive real number or ∞ such that the solution is defined over $[-\tau, t_e)$. Such a $t_e > 0$ exists, because the dynamics (5) grows linearly in x in any bounded open neighborhood of $x(0)$. Later we show that t_e can always be taken to be ∞ for all of the trajectories we are considering. We introduce the operators

$$\begin{array}{l} z(t) = x(t) + \Gamma(t,u_t), \quad \text{where} \\ \Gamma(t,u_t) = \int_{t-\tau}^{t} \lambda(t,m+\tau)B(m+\tau)u(m)\mathrm{d}m \ . \end{array} \tag{19}$$

In all of what follows, we assume that $t \in [0, t_e)$ is arbitrary, unless otherwise noted, and we omit some of the arguments of the time derivatives when they are clear, so $\dot{\Gamma}(t)$ means $(d/dt)\Gamma(t,u_t)$. Then the properties of the fundamental matrix give $\dot{\Gamma}(t) = A(t)\Gamma(t,u_t) + \lambda(t,t+\tau)B(t+\tau)u(t) - B(t)u(t-\tau)$. Using the formula (5) and our decomposition (9) for $F(t,x)$, we obtain

$$\begin{array}{l} \dot{z}(t) = A(t)z(t) \\ +\lambda(t,t+\tau)B(t+\tau)\big[u(t) + f_1(t,\tau,x(t))\big] + f_2(t,x(t)). \end{array} \tag{20}$$

Also, our feedback (18) satisfies $u(t) = -f_1(t,\tau,x(t)) + K(t,\tau)z(t)$. Consequently, in terms of our function H from Assumption 2, (20) becomes

$$\dot{z}(t) = H(t,\tau)z(t) + f_2(t,x(t)) \ . \tag{21}$$

Assumption 2 ensures global asymptotic stability of the linearizations $\dot{z}(t) = H(t,\tau)z(t)$ of (21) at 0. Moreover, the equality

$$x(t) = z(t) - \int_{t-\tau}^{t} \lambda(t,m+\tau)B(m+\tau)u(m)\mathrm{d}m \tag{22}$$

is satisfied.

Second part: decay conditions. We study the stability of the closed loop system using its representation as (21) coupled with (22). We introduce the operator

$$\Xi(u_t) \ = \ \frac{1}{\tau}\int_{t-\tau}^{t} (m - t + 2\tau)|u(m)|\mathrm{d}m \ . \tag{23}$$

Observe for later use that

$$\int_{t-\tau}^{t} |u(m)|\mathrm{d}m \ \le \ \Xi(u_t) \ \le \ 2\int_{t-\tau}^{t} |u(m)|\mathrm{d}m \ . \tag{24}$$

Then, for all $t \ge 0$, we have

$$\dot{\Xi}(t) \ \le \ 2|u(t)| - \tfrac{1}{\tau}\int_{t-\tau}^{t} |u(m)|\mathrm{d}m \ . \tag{25}$$

Also, we can use the upper bound on f_1 from (10), the bound for K given in Assumption 2 and the formula $u(t) = -f_1(t,\tau,x(t)) + K(t,\tau)z(t)$ to get $|u(t)| \le k(\tau)|z(t)| + |x(t)|^2\alpha_1(\tau,|x(t)|^2)$. Moreover, (8) implies that for all $t \ge 0$ and all $z \in \mathbb{R}^n$, we have $q_2(\tau)|z|^2 \le z^\top Q(t,\tau)z$. Taking square roots of both sides of the preceding inequality and the dividing by $\sqrt{q_2(\tau)} > 0$ gives

$$|z| \ \le \ \frac{1}{\sqrt{q_2(\tau)}}\sqrt{R(t,\tau,z)} \ . \tag{26}$$

Combining the last two estimates with (25) gives

$$\begin{array}{rcl} \dot{\Xi}(t) & \le & -\frac{1}{\tau}\int_{t-\tau}^{t} |u(m)|\mathrm{d}m + \frac{2k(\tau)}{\sqrt{q_2(\tau)}}\sqrt{R(t,\tau,z(t))} \\ & & +2|x(t)|^2\alpha_1(\tau,|x(t)|^2) \ . \end{array} \tag{27}$$

We deduce from Assumptions 2-3 that the time derivative of R along all trajectories of (21) satisfies

$$\begin{array}{rcl} \dot{R}(t) & \le & -q_1(\tau)R(t,\tau,z(t)) + 2z(t)^\top Q(t,\tau)f_2(t,x(t)) \\ & \le & -q_1(\tau)R(t,\tau,z(t)) + 2|z(t)|q_3(\tau)|f_2(t,x(t))|. \end{array} \tag{28}$$

From (26), we deduce that

$$\begin{array}{rcl} \dot{R}(t) & \le & -q_1(\tau)R(t,\tau,z(t)) \\ & & +2q_3(\tau)\frac{\sqrt{R(t,\tau,z(t))}}{\sqrt{q_2(\tau)}}|x(t)|^2\alpha_2(|x(t)|^2) \ . \end{array} \tag{29}$$

Consider the family of functions $S_\varepsilon(t,\tau,z) = \sqrt{R(t,\tau,z) + \varepsilon} - \sqrt{\varepsilon}$ parameterized by the constant $\varepsilon \in [0,1)$ and let $S = S_0$. Since R is of class C^1 with respect to t and z, it follows that for all $\varepsilon \in (0,1)$, the function S_ε is of class C^1 with respect to t and z, while S is only continuous. Also, (29) and Lemma 1 in the appendix (applied with the choice $r = R(t,\tau,z)$) give

$$\begin{array}{rcl} \dot{S}_\varepsilon(t) & \le & -q_1(\tau)\frac{R(t,\tau,z(t))}{2\sqrt{R(t,\tau,z(t))+\varepsilon}} \\ & & +\, q_3(\tau)\frac{\sqrt{R(t,\tau,z(t))}|x(t)|^2\alpha_2(|x(t)|^2)}{\sqrt{R(t,\tau,z(t))+\varepsilon}\sqrt{q_2(\tau)}} \\ & \le & -\frac{q_1(\tau)}{2}S(t,\tau,z(t)) \\ & & +\frac{q_3(\tau)}{\sqrt{q_2(\tau)}}|x(t)|^2\alpha_2\left(|x(t)|^2\right) \\ & & +\frac{q_1(\tau)}{2}\varepsilon^{1/4}\Big[1 + S(t,\tau,z(t))\Big] \end{array} \tag{30}$$

along all trajectories of (21).

Third part: Lyapunov-Krasovskii functionals. Let us consider the family of functions

$$V_\varepsilon(t,z,u_t) = a\Xi(u_t) + S_\varepsilon(t,\tau,z) \tag{31}$$

where the constant a satisfies (14) and we omit the argument τ in V_ϵ to simplify the notation. Then, (27) and (30) give

$$\begin{array}{rcl} \dot{V}_\varepsilon(t) & \leq & \left(\frac{2ak(\tau)}{\sqrt{q_2(\tau)}} - \frac{q_1(\tau)}{2}\right) S(t,\tau,z(t)) \\ & & + \frac{q_3(\tau)}{\sqrt{q_2(\tau)}}|x(t)|^2\alpha_2\left(|x(t)|^2\right) \\ & & -\frac{a}{\tau}\int_{t-\tau}^t |u(m)|\mathrm{d}m + 2a|x(t)|^2\alpha_1(\tau,|x(t)|^2) \\ & & +\frac{q_1(\tau)}{2}\varepsilon^{1/4}\left[1 + S(t,\tau,z(t))\right] . \end{array} \tag{32}$$

Since a satisfies (14), we get

$$\begin{array}{l} \dot{V}_\varepsilon(t) \leq \\ -\frac{q_1(\tau)}{4}S(t,\tau,z(t)) + |x(t)|^2\alpha_3(\tau,|x(t)|^2) \\ -\frac{a}{\tau}\int_{t-\tau}^t |u(m)|\mathrm{d}m + \frac{q_1(\tau)}{2}\varepsilon^{1/4}\left[1 + S(t,\tau,z(t))\right] , \end{array} \tag{33}$$

where α_3 was defined in (13).

Next, we find a suitable upper bound on the term $|x(t)|^2\alpha_3(\tau,|x(t)|^2)$ from (33). Our formula (22) for $x(t)$, Assumption 1, and our bound (26) on $|z|$ give

$$\begin{array}{rcl} |x(t)| & \leq & |z(t)| + h(\tau)\int_{t-\tau}^t |u(m)|\mathrm{d}m \\ & \leq & \frac{1}{\sqrt{q_2(\tau)}}S(t,\tau,z(t)) + h(\tau)\int_{t-\tau}^t |u(m)|\mathrm{d}m . \end{array} \tag{34}$$

Recall that our monotonicity properties of β_3 and β_4 from Assumption 3 imply that $m\alpha_3(\tau,m^2)$ is strictly increasing as a function of m for each τ. Therefore, by separately considering the cases where $S(t,\tau,z(t))/\sqrt{q_2(\tau)} \leq h(\tau)\int_{t-\tau}^t |u(m)|\mathrm{d}m$ and where the reverse inequality holds, we get

$$\begin{array}{l} |x(t)|^2\alpha_3(\tau,|x(t)|^2) \leq \\ \frac{4}{q_2(\tau)}S^2(t,\tau,z(t))\alpha_3\left(\tau, \frac{4}{q_2(\tau)}S^2(t,\tau,z(t))\right) \\ +4h^2(\tau)\left[\int_{t-\tau}^t |u(m)|\mathrm{d}m\right]^2 \\ \quad \times\alpha_3\left(\tau, 4h^2(\tau)\left[\int_{t-\tau}^t |u(m)|\mathrm{d}m\right]^2\right) . \end{array} \tag{35}$$

We can combine this inequality with (33) to get

$$\begin{array}{l} \dot{V}_\varepsilon(t) \leq \Big[-\frac{q_1(\tau)}{4} \\ +\frac{4}{q_2(\tau)}S(t,\tau,z(t))\alpha_3\left(\tau,\frac{4}{q_2(\tau)}S^2(t,\tau,z(t))\right)\Big] S(t,\tau,z(t)) \\ +\Bigg[-\frac{a}{\tau} \\ +4h^2(\tau)\int_{t-\tau}^t |u(m)|\mathrm{d}m\,\alpha_3\left(\tau, 4h^2(\tau)\left[\int_{t-\tau}^t |u(m)|\mathrm{d}m\right]^2\right)\Bigg] \\ \quad \times\int_{t-\tau}^t |u(m)|\mathrm{d}m + \frac{q_1(\tau)}{2}\varepsilon^{1/4}\left[1 + S(t,\tau,z(t))\right] . \end{array}$$

Also, $V_\varepsilon(t,z(t),u_t) \geq \sqrt{R(t,\tau,z(t))+\varepsilon} - \sqrt{\varepsilon} \geq S(t,\tau,z(t)) - \sqrt{\varepsilon}$ and $V_\varepsilon(t,z(t),u_t) \geq a\int_{t-\tau}^t |u(m)|\mathrm{d}m$ hold for all $\varepsilon \in [0,1)$, by (24). Since $m\alpha_3(\tau,m^2)$ is increasing in m for each τ, it follows that

$$\begin{array}{l} S(t,\tau,z(t))\alpha_3\left(\tau,\frac{4}{q_2(\tau)}S^2(t,\tau,z(t))\right) \leq \\ \left[V_\varepsilon(t,z(t),u_t) + \sqrt{\varepsilon}\right]\alpha_3\left(\tau, \frac{4}{q_2(\tau)}\left[2V_\varepsilon(t,z(t),u_t)\sqrt{\varepsilon} + \varepsilon\right]\right. \\ \left. +\frac{4}{q_2(\tau)}V_\varepsilon^2(t,z(t),u_t)\right) . \end{array}$$

We now apply (11)-(12), with $b = \frac{4}{q_2(\tau)}\left(2V_\varepsilon(t,z(t),u_t)\sqrt{\varepsilon} + \varepsilon\right)$ and $c = 4V_\varepsilon^2(t,z(t),u_t)/q_2(\tau)$, and use the fact that $\varepsilon \leq \sqrt{\varepsilon} \leq \varepsilon^{1/4}$ for all $\varepsilon \in [0,1]$, to find a continuous positive valued and nondecreasing function φ_c (also depending on τ, but independent of ϵ) such that

$$\begin{array}{rcl} \dot{V}_\varepsilon(t) & \leq & \Big[-\frac{q_1(\tau)}{4} + \frac{4}{q_2(\tau)}V_\varepsilon(t,z(t),u_t) \\ & & \times\alpha_3\left(\tau,\frac{4}{q_2(\tau)}V_\varepsilon^2(t,z(t),u_t)\right)\Big] S(t,\tau,z(t)) \\ & & +\frac{1}{a}\Big[-\frac{a^2}{\tau} + 4h^2(\tau)V_\varepsilon(t,z(t),u_t) \\ & & \times\alpha_3\left(\tau, 4h^2(\tau)\frac{V_\varepsilon^2(t,z(t),u_t)}{a^2}\right)\Big]\int_{t-\tau}^t |u(m)|\mathrm{d}m \\ & & +\varepsilon^{1/4}\varphi_c(V_\varepsilon(t,z(t),u_t)) . \end{array} \tag{36}$$

Next, recall that our assumption (17) implies that $\sqrt{q_3(\tau)}|z(0)| + \frac{a}{\tau}\int_{-\tau}^0 (m+2\tau)|u(m)|\mathrm{d}m < v(\tau)$, where $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ as before. Then (8) from Assumption 2 gives $V_0(0,z(0),u_0) < v(\tau)$. Since $\sqrt{c_1+c_2} \leq \sqrt{c_1} + \sqrt{c_2}$ holds for all nonnegative constants c_1 and c_2, we know that $V_\varepsilon \leq V_0$ holds pointwise for all $\varepsilon \in (0,1]$. It follows that $V_\varepsilon(0,z(0),u_0) \leq V_0(0,z(0),u_0) < \overline{v}$ hold for all $\varepsilon \in (0,1]$, where $\overline{v} = [V_0(0,z(0),u_0) + v(\tau)]/2 > 0$. Then $\overline{v} < v(\tau)$.

Set $\overline{v}_a = (v(\tau) + \overline{v})/2$. Since $m\alpha_3(\tau,m^2)$ is strictly increasing in m, and since $\overline{v}_a < v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$, it follows from our conditions (15)-(16) on $v_1(\tau)$ and $v_2(\tau)$ that the constants

$$\begin{array}{rcl} \overline{p}_1 & = & \frac{q_1(\tau)}{4} - \frac{4}{q_2(\tau)}\overline{v}_a\alpha_3\left(\tau,\frac{4}{q_2(\tau)}\overline{v}_a^2\right) \text{ and} \\ \overline{p}_2 & = & \frac{a^2}{\tau} - 4h^2(\tau)\overline{v}_a\alpha_3\left(\tau, 4h^2(\tau)\frac{\overline{v}_a^2}{a^2}\right) \end{array} \tag{37}$$

are positive for all $\tau > 0$. Fix any value of $\varepsilon \in (0,1]$ satisfying

$$\varepsilon \in \left(0, \left(\frac{\min\{\overline{p}_1,\overline{p}_2\}\overline{v}}{4\varphi_c(\overline{v}_a)\max\{a^2,1\}}\right)^4\right], \tag{38}$$

where the left endpoint is omitted because we need $\varepsilon > 0$. Next, we prove by contradiction that $V_\varepsilon(t,z(t),u_t) \leq \overline{v}$ for all $t \geq 0$. Assume that this property does not hold. Then, since $\overline{v}_a > \overline{v}$ and $V_\varepsilon(0,z(0),u_0) < \overline{v}$, we can find a $t_2 > 0$ such that $V_\varepsilon(t,z(t),u_t) \leq \overline{v}_a$ for all $t \in [0,t_2]$ and $V_\varepsilon(t_2,z(t_2),u_{t_2}) > \overline{v}$. Set $t_1 = \inf\{t \leq t_2 : V_\varepsilon(p,z(p),u_p) \geq \overline{v} \text{ for all } p \in [t,t_2]\}$. Then, since $t \mapsto V_\varepsilon(t,z(t),u_t)$ is continuous, we get $V_\varepsilon(t,z(t),u_t) \in [\overline{v},\overline{v}_a]$ for all $t \in [t_1,t_2]$, $V_\varepsilon(t_1,z(t_1),u_{t_1}) = \overline{v}$, and $\dot{V}_\varepsilon(t_1) \geq 0$.

By (36) and the fact that $l\alpha_3(\tau,l^2)$ is strictly increasing in l,

$$\begin{array}{rcl} \dot{V}_\varepsilon(t) & \leq & -\overline{p}_1 S(t,\tau,z(t)) - \frac{1}{a}\overline{p}_2\int_{t-\tau}^t |u(m)|\mathrm{d}m \\ & & +\varepsilon^{1/4}\varphi_c(\overline{v}_a) \end{array} \tag{39}$$

for all $t \in [t_1,t_2]$. It follows from our lower bound on Ξ from (24) that

$$\begin{array}{rcl} \dot{V}_\varepsilon(t) & \leq & -\frac{1}{2\max\{a^2,1\}}\min\{\overline{p}_1,\overline{p}_2\}V_0(t,z(t),u_t) \\ & & +\varepsilon^{1/4}\varphi_c(\overline{v}_a) \\ & \leq & -\frac{1}{2\max\{a^2,1\}}\min\{\overline{p}_1,\overline{p}_2\}V_\varepsilon(t,z(t),u_t) \\ & & +\varepsilon^{1/4}\varphi_c(\overline{v}_a) \end{array} \tag{40}$$

for all $t \in [t_1,t_2]$. Since $V_\varepsilon(t,z(t),u_t) \in [\overline{v},\overline{v}_a]$ for all $t \in [t_1,t_2]$, we deduce that

$$\begin{array}{rcl} \dot{V}_\varepsilon(t) & \leq & -\frac{1}{2\max\{a^2,1\}}\min\{\overline{p}_1,\overline{p}_2\}\overline{v} + \varepsilon^{1/4}\varphi_c(\overline{v}_a) \\ & \leq & -\frac{\min\{\overline{p}_1,\overline{p}_2\}}{4\max\{a^2,1\}}\overline{v} < 0 \end{array} \tag{41}$$

for all $t \in [t_1,t_2]$ when ε satisfies (38). It follows that $\dot{V}_\varepsilon(t_1) < 0$. This yields a contradiction with the choice of t_1. Hence, when (38) holds, we get $V_\varepsilon(t,z(t),u_t) \leq \overline{v}$ for all $t \geq 0$, which implies that we can choose $t_e = \infty$. Also, arguing as we did before, we get

$$\dot{V}_\varepsilon(t) \leq -\frac{\min\{\overline{p}_1,\overline{p}_2\}}{2\max\{a^2,1\}}V_\varepsilon(t,z(t),u_t) + \varepsilon^{1/4}\varphi_c(\overline{v}) . \tag{42}$$

for all $t \geq 0$. This gives a value $t_c > 0$ such that for all $t \geq t_c$, we have

$$V_\varepsilon(t, z(t), u_t) \;\leq\; \frac{4\varphi_c(\overline{v})\varepsilon^{1/4}}{\min\{\overline{p}_1, \overline{p}_2\}} \max\{a^2, 1\} \tag{43}$$

(since V_ε is nonnegative valued), and therefore also

$$\begin{aligned} \Xi(u_t) \;&\leq\; \frac{4\varphi_c(\overline{v})\varepsilon^{1/4}}{a\min\{\overline{p}_1, \overline{p}_2\}} \max\{a^2, 1\} \quad \text{and} \\ S_\varepsilon(t, \tau, z) \;&\leq\; \frac{4\varphi_c(\overline{v})\varepsilon^{1/4}}{\min\{\overline{p}_1, \overline{p}_2\}} \max\{a^2, 1\}\;. \end{aligned} \tag{44}$$

Since $S_\varepsilon(t, \tau, z) = \sqrt{R(t, \tau, z) + \varepsilon} - \sqrt{\varepsilon} \geq \sqrt{q_2(\tau)}|z| - \sqrt{\varepsilon}$ holds for all t, τ, and z, (24) gives

$$\begin{aligned} &\max\left\{\int_{t-\tau}^{t} |u(m)|\mathrm{d}m, \sqrt{q_2(\tau)}|z|\right\} \;\leq \\ &\sqrt{\varepsilon} + \tfrac{4\varphi_c(\overline{v})\varepsilon^{1/4}}{\min\{a,1\}\min\{\overline{p}_1,\overline{p}_2\}} \max\{a^2,1\} \end{aligned} \tag{45}$$

for all $t \geq t_c$. Set

$$\begin{aligned} &\Delta = \\ &\max\left\{\tfrac{1}{\sqrt{q_2(\tau)}}, 1\right\}\left(1 + \tfrac{4\varphi_c(\overline{v})}{\min\{a,1\}\min\{\overline{p}_1,\overline{p}_2\}} \max\{a^2, 1\}\right). \end{aligned} \tag{46}$$

Then, since $\varepsilon \in [0, 1]$, it follows that for all $t \geq t_c$, the inequalities

$$|z(t)| \;\leq\; \Delta\varepsilon^{1/4} \quad \text{and} \quad \int_{t-\tau}^{t} |u(m)|\mathrm{d}m \;\leq\; \Delta\varepsilon^{1/4} \tag{47}$$

are satisfied. Since ε is arbitrarily small, we deduce that $|z(t)|$ and $\int_{t-\tau}^{t} |u(m)|\mathrm{d}m$ converge to zero when $t \to \infty$. This and the first inequality in (34) imply that $x(t) \to 0$ as $t \to \infty$. Also, by letting ε depend on the maximum of V_0 on a suitable neighborhood of the origin, we can prove the local stability part. This proves the theorem.

VI. Arbitrarily Large Domains of Attraction

Theorem 1 applies for all $\tau > 0$. On the other hand, consider the special case where $f_2 = 0$ in the decomposition (9) of F. Then, setting $\tau = 0$ in (9) and in our control (18) produces the uniformly globally asymptotically stable closed loop system $\dot{x}(t) = [A(t) + B(t)K(t, 0)]x(t)$ from Assumption 2. This suggests that the domain of attraction should become arbitrarily large as $\tau \to 0^+$ when $f_2 = 0$. Our next theorem implies that this is indeed the case. We will assume that the functions q_i and k from Assumption 2 are constant, so we omit their arguments τ. This is not restrictive, since now we only need to consider τ's on a bounded interval; see Remark 1.

Corollary 1. *Let Assumptions 1-3 hold with $f_2 = 0$ and the q_i's and k all constant. Then for each constant $v_* > 0$, we can find values $a \in (0, q_1\sqrt{q_2}/(8k))$ and $\tau_M > 0$ (both depending on v_*) such that: For each initial condition $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$ satisfying*

$$\begin{aligned} &\sqrt{q_3(\tau)}\left|\phi_x(0) + \int_{-\tau}^{0} \lambda(0, m+\tau)B(m+\tau)\phi_u(m)\mathrm{d}m\right| \\ &+ \tfrac{a}{\tau}\int_{-\tau}^{0} (m + 2\tau)|\phi_u(m)|\mathrm{d}m < v_* \end{aligned} \tag{48}$$

and each constant delay $\tau \in (0, \tau_M)$, the trajectory of (5) in closed loop with (18) converges to 0 as $t \to \infty$. □

Proof. We set $\alpha_2 = 0$, so we have $\alpha_3 = 2a\alpha_1$. Then (15)-(16) become

$$\begin{aligned} v_1(\tau)\alpha_1\left(\tau, \tfrac{4}{q_2}v_1^2(\tau)\right) &= \tfrac{q_1 q_2}{32a} \quad \text{and} \\ v_2(\tau)\alpha_1\left(\tau, \tfrac{4h^2(\tau)}{a^2}v_2^2(\tau)\right) &= \tfrac{a}{8\tau h^2(\tau)}\;. \end{aligned} \tag{49}$$

For each constant $\tau_M > 0$, Assumption 3 provides a function $\overline{\gamma}$ of class $\mathcal{K}_\infty$ such that $m\alpha_1\left(\tau, m^2\right) \leq \overline{\gamma}(m)$ for all $\tau \in [0, \tau_M]$ and $m \geq 0$. Then, replacing $\alpha_1(\tau, m^2)$ in (49) by $\overline{\gamma}(m)/m$ gives

$$\begin{aligned} \overline{\gamma}\left(\sqrt{\tfrac{4}{q_2}}v_1(\tau)\right) &= \tfrac{q_1\sqrt{q_2}}{16a} \quad \text{and} \\ \overline{\gamma}\left(\tfrac{2h(\tau)}{a}v_2(\tau)\right) &= \tfrac{1}{4\tau h(\tau)} \end{aligned} \tag{50}$$

for all $\tau \in (0, \tau_M)$. Our proof of Theorem 1 shows that the conclusions of that theorem remain true when $v_1(\tau)$ and $v_2(\tau)$ are defined to be the solutions of (50). Therefore,

$$\begin{aligned} v_1(\tau) &= \tfrac{\sqrt{q_2}}{2}\overline{\gamma}^{-1}\left(\tfrac{q_1\sqrt{q_2}}{16a}\right) \quad \text{and} \\ v_2(\tau) &= \tfrac{a}{2h(\tau)}\overline{\gamma}^{-1}\left(\tfrac{1}{4\tau h(\tau)}\right)\;. \end{aligned} \tag{51}$$

Also, when τ is sufficiently small, the choice

$$a = 1\Bigg/\sqrt{\overline{\gamma}^{-1}\left(\frac{1}{4\tau h(\tau)}\right)} \tag{52}$$

will satisfy our requirements (14) on a, because (52) converges to 0 as $\tau \to 0^+$ and because we are now assuming that the q_i's and k are positive constants. Then (51) become

$$\begin{aligned} v_1(\tau) &= \tfrac{\sqrt{q_2}}{2}\overline{\gamma}^{-1}\left(\tfrac{q_1\sqrt{q_2}}{16}\sqrt{\overline{\gamma}^{-1}\left(\tfrac{1}{4\tau h(\tau)}\right)}\right) \text{ and} \\ v_2(\tau) &= \tfrac{1}{2h(\tau)}\sqrt{\overline{\gamma}^{-1}\left(\tfrac{1}{4\tau h(\tau)}\right)}\;. \end{aligned} \tag{53}$$

Therefore, both $v_1(\tau)$ and $v_2(\tau)$ converge to ∞ when $\tau \to 0^+$. It follows that $v(\tau) \to \infty$ as $\tau \to 0^+$, so we can satisfy (48) for small enough $\tau > 0$ by choosing τ such that $v(\tau) > v_*$. The corollary now follows from Theorem 1. □

VII. Illustrative Example

We illustrate Theorem 1 using the 1 dimensional system from (7), which is

$$\dot{x}(t) = x(t) + u(t - \tau) + lx^2(t)\sin(x(t)) \tag{54}$$

where $u \in \mathbb{R}$ is the input, τ is a positive constant delay, and l is a positive constant. This system is not globally Lipschitz in the state x. With the notation of the previous sections, we have $A = 1$, $B = 1$, $\lambda(t, t_0) = e^{t-t_0}$, and $F(t, x) = lx^2\sin(x)$. As we saw in Section IV, (54) satisfies our assumptions with $h(\tau) = 1$, $K(t, \tau) = -2e^\tau$, $Q(t, \tau) = \frac{1}{2}$, $q_1(\tau) = 2$, $q_2(\tau) = q_3(\tau) = \frac{1}{2}$, $k(\tau) = 2e^\tau$, $f_2 = 0$, $f_1(t, \tau, x) = le^\tau x^2\sin(x)$, $\alpha_1(\tau, m) = le^\tau$ and $\alpha_2(m) = 0$. According to (14), the inequalities $0 < a \leq 1/(8\sqrt{2}e^\tau)$ have to be satisfied and, by the expression of α_3 in (13), $\alpha_3(\tau, m) = 2ale^\tau$.

Choosing

$$a = \frac{1}{8\sqrt{2}e^\tau}, \tag{55}$$

we can straightforwardly derive an estimate of the basin of attraction from Theorem 1 by using $v = \min\{v_1, v_2\}$, where

$$v_1(\tau) = \frac{1}{2\sqrt{2}l} \tag{56}$$

and

$$v_2(\tau) = \frac{1}{64\sqrt{2}\tau e^{2\tau}l}, \tag{57}$$

which converge to ∞ as $l \to 0$ for each $\tau > 0$. On the other hand, when $\tau \in (0, 1]$, we can take

$$a = \frac{\sqrt{\tau}}{8\sqrt{2}e^\tau} \tag{58}$$

to obtain the values

$$v_1(\tau) = \frac{1}{2l\sqrt{2\tau}} \tag{59}$$

and

$$v_2(\tau) = \frac{1}{64le^{2\tau}\sqrt{2\tau}}, \quad (60)$$

so $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ converges to ∞ as l converges to zero for fixed $\tau > 0$, or as τ converges to zero for fixed l, so the basin of attraction becomes arbitrarily large. This gives convergence of the closed loop solution to 0.

If, on the other hand, we had chosen, $f_1 = 0$ and $f_2(t,x) = lx^2\sin(x)$, then one could choose $\alpha_1 = c_0$ for any constant $c_0 > 0$ and $\alpha_2(m) = l$. This gives $\alpha_3(\tau, m) = 2ac_0 + \frac{1}{\sqrt{2}}l$. Then the corresponding solutions of (15)-(16) with the choice

$$a = \frac{1}{8\sqrt{2}e^{\tau}} \quad (61)$$

satisfy

$$v_1(\tau) \leq \frac{\sqrt{2}}{16l} \quad (62)$$

and

$$v_2(\tau) \leq \frac{1}{256\sqrt{2}e^{2\tau}\tau l}, \quad (63)$$

which would mean that $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ does not converge to ∞ as τ goes to zero. Thus, the choice $f_1 = 0$ and $f_2(t,x) = lx^2\sin(x)$ is conservative.

VIII. Conclusions

Stabilization of nonlinear systems with input delays is a central problem that has been studied by many authors using model reduction, prediction, and other methods. Here we adapted the reduction model approach to the problem of locally asymptotically stabilizing the origin of time varying nonlinear systems with pointwise input delays. Our method of proof makes it possible to determine an estimate of the basin of attraction. The result can be adapted to the case where the delay in the input is distributed. Our results can be combined with those of [10] and [5].

Appendix: Technical lemma

We used the following to get (30) in the second part of the proof of Theorem 1:

Lemma 1. *Let $\varepsilon \in (0,1]$ be a positive real number. Then*

$$-\frac{r}{\sqrt{r+\varepsilon}} \leq -\sqrt{r} + \varepsilon^{1/4}[1+\sqrt{r}] \quad (64)$$

holds for all $r \geq 0$.

Proof. Let $r \geq 0$ be given. We first prove that

$$\frac{r}{\sqrt{r+\varepsilon}} \geq \frac{1}{\sqrt{1+\sqrt{\varepsilon}}}\sqrt{r} - \varepsilon^{1/4}. \quad (65)$$

If $\sqrt{r}/(\sqrt{1+\sqrt{\varepsilon}}) - \varepsilon^{1/4} \leq 0$, then (65) is satisfied. On the other hand, if $\sqrt{r}/(\sqrt{1+\sqrt{\varepsilon}}) - \varepsilon^{1/4} \geq 0$, then $r \geq (1+\sqrt{\varepsilon})\sqrt{\varepsilon}$. It follows that $(\sqrt{\varepsilon}+1)r \geq (1+\sqrt{\varepsilon})\varepsilon + r \geq \varepsilon + r$. Consequently, $r/(r+\varepsilon) \geq 1/(\sqrt{\varepsilon}+1)$. Taking the square root, and then multiplying through by $\sqrt{r}$, we obtain

$$r\sqrt{\frac{1}{r+\varepsilon}} \geq \frac{\sqrt{r}}{\sqrt{\sqrt{\varepsilon}+1}}. \quad (66)$$

Therefore, (65) holds in both cases. Next, observe that (65) implies that

$$\begin{aligned} -\frac{r}{\sqrt{r+\varepsilon}} &\leq -\sqrt{r} + \left[1 - \frac{1}{\sqrt{1+\sqrt{\varepsilon}}}\right]\sqrt{r} + \varepsilon^{1/4} \\ &\leq -\sqrt{r} + \left[\sqrt{1+\sqrt{\varepsilon}} - 1\right]\sqrt{r} + \varepsilon^{1/4}\ . \end{aligned} \quad (67)$$

Hence, the relation $\sqrt{b+c} \leq \sqrt{b} + \sqrt{c}$ for nonnegative values b and c gives $-r/\sqrt{r+\varepsilon} \leq -\sqrt{r} + \varepsilon^{1/4}\sqrt{r} + \varepsilon^{1/4}$. This gives the conclusion. □

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