Lyapunov Technique and Backstepping for Nonlinear Neutral Systems

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***Abstract*—For nonlinear systems with delay of neutral type, we propose a new technique of stability and robustness analysis. It relies on the construction of functionals which make it possible to establish estimates of the solutions different from, but very similar to, estimates of ISS or iISS type. These functionals are themselves different from, but very similar to, ISS or iISS Lyapunov-Krasovskii functionals. The approach applies to systems which do not have a globally Lipschitz vector field and are not necessarily locally exponentially stable. We apply this technique to carry out a backstepping design of stabilizing control laws for a family of neutral nonlinear systems.**

I. Introduction

Research devoted to systems with delay shows a lot of interest in systems of neutral type. Indeed, these systems are encountered in a wide range of engineering problems which include heat exchangers [14], population ecology [15, Chapt. 2], distributed networks [5]. These systems have the feature of incorporating retarded derivatives of state variable. Typically they are of the type

$$\dot{x}(t) = \mathcal{F}(\dot{x}(t-\tau), x(t), x(t-\tau)), \qquad (1)$$

where $x \in \mathbb{R}^n$, where $\tau > 0$ is a constant delay [27], but they can be more complicated: in particular they may be time-varying, include several pointwise delays, distributed or time-varying delays and disturbances.

Many papers have explored many problems for these systems in the particular case where the function $\mathcal{F}$ is linear. In some contributions, the frequency domain approach has been used to provided with sufficient stability conditions: see [4], [19], [9]. In the time-domain, LMI conditions ensuring the asymptotic stability of the origin of the studied systems have been proposed in [22], [6], [8], [3], [7] and in other papers. Further studies have extended these conditions to cases where the function $\mathcal{F}$ is nonlinear. In most of them, the nonlinearities are regarded as disturbances and the stability of the system is deduced from the presence of stabilizing linear terms: results of this type, for instance, are given in [25], [13], [27]. A common feature of these works is that they apply only to systems that are locally exponentially stable and globally Lipschitz.

The results presented in [15, Chapt. 9] are different from all those we have mentioned: the local asymptotic stability of a family of nonlinear systems, which are not necessarily locally exponentially stable, is established by using a Lyapunov functional. The present study owes a great deal to [15, Chapt. 9]: the first main objective we pursue is to complement [15, Chapt. 9] in two directions: (i) we aim at establishing results of global asymptotic stability (ii) we wish to construct Lyapunov-Krasovskii functionals from which robustness properties can be derived. To achieve these goals, we propose a new technique of stability and robustness analysis for nonlinear neutral systems based on the design of functionals of a new type. The family of systems to which our technique applies is general in the sense that it includes systems that are not globally Lipschitz and not locally exponentially stable. The robustness result we establish is different, but very similar to the ISS or iISS robustness. Our technique of proof relies on the construction of functionals which are not Lyapunov-Krasovskii functionals as defined in preceding studies, for instance in [10], [16], [21], [12] or iISS Lyapunov-Krasovskii functionals as defined for instance in [20], [11], but they are very similar to these functionals. Our approach to neutral systems also differs from the analysis method of Lyapunov-Krasovskii type developed in [21] in the sense that this paper does not require the formulation into coupled delay differential and difference equations with an auxiliary variable in considering the robustness. This first robust stability result in this paper will help us to achieve our second goal, which is the extension of the backstepping approach to nonlinear neutral systems. Thus, we will complement that way the backstepping result for systems with delay presented in [17], [18] and [2]. To the best of our knowledge, the result we will propose is the first backstepping result for neutral systems and it is not covered by any of the scarce results of construction of stabilizing feedbacks for nonlinear neutral systems available in the literature, as for instance those in [24], [26]. For the sake of simplicity, we have considered the case of systems with only one input, but extensions to the multi-input case can be easily deduced from our design.

Our paper is organized as follows. We state and prove the main result in Section II. We present a backstepping design in Section III. Some applications of the theoretical results are given in Section IV. Concluding remarks are drawn in Section V.

Notation and definitions. • We let $|\cdot|$ denote the Euclidean norm of matrices and vectors of any dimension. • Given $\phi : \mathcal{I} \to \mathbb{R}^p$ defined on an interval $\mathcal{I} \subset \mathbb{R}$, let $|\phi|_{\mathcal{I}}$ denote its (essential) supremum over $\mathcal{I}$. • A function $\psi : [a, b) \to \mathbb{R}^k$, where a is a real numbers and $b > a$ is a real number or $+\infty$, is piecewise continuously differentiable if it is continuous on the interval $[a, b)$ and, for any real number $c \in [a, b)$, $\dot{\psi}$ is

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continuous over $[a,c]$, except at a finite number of points. • We let $C_{\rm in}$ denote the set of all $\mathbb{R}^n$-valued functions ϕ defined on a given interval $[-\tau,0]$ that are continuous, piecewise continuously differentiable and with an essentially bounded first derivative $\dot{\phi}$. • For a function $x : [-\tau,T) \to \mathbb{R}^k$ with $T=+\infty$ or $T \in (0,+\infty)$, for all $t \in [0,T)$, the function x_t is defined by $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-\tau,0]$. The solution of a time-delay system described by a functional differential equation:

$$\dot{\mathcal{X}}(t) = \mathcal{F}\left(\dot{\mathcal{X}}(t-\tau), \mathcal{X}_t, w(t)\right), \qquad (2)$$

where w is a continuous function, with an initial condition $\phi_{\mathcal{X}} \in C_{\rm in}$ at $t_0=0$, will be denoted by $\mathcal{X}(t)$ (instead of $\mathcal{X}(t,t_0,\phi_\chi)$ as rigorously done for instance in [10, Chapt. 2]). • The notation will be simplified whenever no confusion can arise from the context. In particular, the time derivative of a Lyapunov functional $V(\mathcal{X}_t)$ along the trajectories of a system (2) will be denoted simply by $\dot{V}(t)$. • A continuous function $\gamma : [0,\infty) \to [0,\infty)$ belongs to class $\mathcal{K}$ provided that it is strictly increasing and $\gamma(0)=0$. It belongs to class $\mathcal{K}_\infty$ if, in addition, $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\mathcal{KL}$ provided that for each fixed $s \geq 0$, the function $\beta(\cdot,s)$ belongs to class $\mathcal{K}$, and for each fixed $r \geq 0$, the function $\beta(r,\cdot)$ is non-increasing and $\beta(r,s) \to 0$ as $s \to 0$. A class $\mathcal{K}$ function γ is said to be equal to a quadratic function in a neighborhood of the origin if there exist two constants $c_1>0$, $c_2>0$ such that $\gamma(r) = c_1 r^2$ for all $r \in [0,c_2]$.

II. General result for neutral systems

We consider a functional differential equation

$$\begin{aligned} \dot{x}(t) &= F(\dot{x}(t-\tau), x(t), \delta(t)), \\ x_0 &= \phi(t),\ \dot{x}_0 = \dot{\phi}(t) \end{aligned} \qquad (3)$$

with $x(t) \in \mathbb{R}^n$, with initial conditions ϕ in $C_{\rm in}$ at $t=0$, with $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ locally Lipschitz, with $\delta : [0,+\infty) \to \mathbb{R}^p$ continuous and $\tau>0$ is a constant delay.

We introduce two assumptions. Generally speaking, the first one imposes a limitation of growth type on the dependency of F with respect to $\dot{x}(t-\tau)$ and the second one also limits the influence of $\dot{x}(t-\tau)$ and guarantees the existence of a Lyapunov function whose derivative along the solutions of the studied system satisfies an inequality of iISS type when both $\dot{x}(t-\tau)$ and $\delta(t)$ are regarded as inputs.

Assumption 1. *There exist a nonnegative real number ε, a continuous function $\varpi : \mathbb{R}^n \to [0,+\infty)$ and a function α_1 of class $\mathcal{K}$ such that for all $k \in \mathbb{R}^n$, $l \in \mathbb{R}^n$, $m \in \mathbb{R}^p$,*

$$|F(k,l,m)| \leq \varepsilon|k| + \varpi(l) + \alpha_1(|m|). \qquad (4)$$

Assumption 2. *There are a function μ of class C^1, two functions κ_1 and κ_2 of class $\mathcal{K}_\infty$, a continuous positive definite function ν and two functions ψ and α_2 of class $\mathcal{K}$ so that, for all $x \in \mathbb{R}^n$,*

$$\kappa_1(|x|) \leq \mu(x) \leq \kappa_2(|x|) \qquad (5)$$

and for all $(a,b,\Delta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$,

$$\frac{\partial\mu}{\partial x}(b) F(a,b,\Delta) \leq -\nu(b) + \psi(|a|) + \alpha_2(|\Delta|). \qquad (6)$$

Moreover, for all $\ell \geq 0$,

$$3\psi(2\varepsilon\ell) \leq \psi(\ell), \qquad (7)$$

where ε is the constant provided by Assumption 1, for all $x \in \mathbb{R}^n$,

$$3\psi(4\varpi(x)) \leq \nu(x), \qquad (8)$$

where ϖ is the function provided by Assumption 1, and there exists a function ρ of class $\mathcal{K}$ such that, for all $x \in \mathbb{R}^n$,

$$\rho(\mu(x)) \leq \nu(x). \qquad (9)$$

We are ready to state and prove the main result of the paper.

Theorem 1: Assume that the system (3) satisfies Assumptions 1 and 2. Let $c>0$ be a positive real number such that the inequality

$$c \leq 1 - \frac{2e^{c\tau}}{3} \qquad (10)$$

is satisfied. Then the derivative of the functional $U : C_{\rm in} \to [0,+\infty)$,

$$U(\phi_x) = \mu(\phi_x(0)) + 2\int_{-\tau}^{0} e^{c(\ell+\tau)}\psi\left(|\dot{\phi}_x(\ell)|\right) d\ell \qquad (11)$$

along the trajectories of (3) satisfies

$$\dot{U}(t) \leq -\overline{\rho}(U(x_t)) - c\psi(|\dot{x}(t-\tau)|) + \alpha_3(|\delta(t)|) \qquad (12)$$

for almost all $t \in [0,+\infty)$, with

$$\overline{\rho}(m) = \frac{c}{2}\min\{m, \rho(m)\},\ \forall m \geq 0 \qquad (13)$$

and

$$\alpha_3(m) = \alpha_2(m) + 2e^{c\tau}\psi\left(4\alpha_1(m)\right),\ \forall m \geq 0. \qquad (14)$$

Discussion on Theorem 1.

• It is worth noting that for any $\tau>0$, one can determine a constant $c>0$ such that the inequality (10) is satisfied. Therefore the asymptotic stability of the system (3) with δ identically equal to zero which is implied by Theorem 1 is delay independent. Notice that (10) is satisfied for all $c \in \left(0, \min\left\{\frac{1}{6}, \frac{1}{\tau}\ln\left(\frac{5}{4}\right)\right\}\right]$.

• The requirement (7) imposes a restriction on the size of ε. This assumption can possibly be relaxed, but it cannot be removed: some systems which satisfy Assumptions 1 and 2 except (7) are not locally asymptotically stable. For instance the one-dimensional system $\dot{x}(t) = 2\dot{x}(t-\tau) - x(t)$ is exponentially unstable when τ is smaller than a certain threshold.

• A remarkable feature of Theorem 1 is that it applies to systems that are not necessarily locally exponentially stable or globally Lipschitz.

• The family of functionals U in (11) and the ones of [15, Chapt. 9] differ mostly because each functional U admits a derivative along the solutions of the system studied that is smaller than a negative definite function of $U(x_t)$, which is not the case of the functionals provided in [15, Chapt. 9]. The advantage of knowing functionals satisfying (12) is that they lead to robustness properties: from the features of $\overline{\rho}$ and U, and by adapting the proofs of [23], [1] (see also [16, Chapt. 2]), one can deduce from (12) that all the solutions of (3) are defined over $[-\tau, +\infty)$ and there are a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for any solution $x(t)$ of (3), for all times $t \geq 0$, the inequality

$$\begin{array}{rcl} |x(t)| & \leq & \beta(|x(0)| + \sup\limits_{m\in[-\tau,0]} \{|\dot{x}(m)|\}, t) \\ & & +\gamma(\int_0^t |\delta(\ell)|d\ell) \end{array} \tag{15}$$

holds. Moreover, if the function ρ provided by Assumption 2 is of class $\mathcal{K}_\infty$, there are a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for any solution $x(t)$ of (3), for all times t, $t \geq 0$, the inequality

$$\begin{array}{rcl} |x(t)| & \leq & \beta(|x(0)| + \sup\limits_{m\in[-\tau,0]} \{|\dot{x}(m)|\}, t) \\ & & +\gamma(\sup_{m\in[0,t]} |\delta(m)|) \end{array} \tag{16}$$

holds. In both cases, the functions β and γ are delay-dependent.

Proof. Theorem 3.3 in [15, Chapt. 3] ensures that for any initial condition $\phi \in C_{\text{in}}$, the system (3) admits a unique solution x, which is piecewise continuously differentiable and defined over the interval $[-\tau, t_\phi)$, where t_ϕ is a finite positive real number or $+\infty$. If the maximum of such intervals satisfies $t_\phi < +\infty$, then $\dot{x}$ is not essentially bounded over $[-\tau, t_\phi)$.

Now, we introduce the operator $\Gamma : C_{\text{in}} \to [0, +\infty)$ defined by

$$\Gamma(\phi) = 2\int_{-\tau}^{0} e^{c(\ell+\tau)}\psi\left(|\dot{\phi}(\ell)|\right)d\ell, \tag{17}$$

where $c > 0$ is a constant such that (10) holds. It satisfies, along the trajectories of (3),

$$\Gamma(x_t) = 2\int_{t-\tau}^{t} e^{-c(t-\ell-\tau)}\psi(|\dot{x}(\ell)|)d\ell. \tag{18}$$

In the sequel, all equalities and inequalities of $\dot{\Gamma}(t)$ and $\dot{U}(t)$ should be understood to hold for all $t \in [0, t_\phi)$, *almost everywhere*. Elementary calculations yield

$$\dot{\Gamma}(t) = -c\Gamma(x_t) + 2e^{c\tau}\psi(|\dot{x}(t)|) - 2\psi(|\dot{x}(t-\tau)|). \tag{19}$$

Since U, defined in (11), satisfies, along the trajectories of (3),

$$U(x_t) = \mu(x(t)) + \Gamma(x_t), \tag{20}$$

by (6) in Assumption 2 and (20) we obtain

$$\begin{array}{c} \dot{U}(t) \leq -\nu(x(t)) + \psi(|\dot{x}(t-\tau)|) + \alpha_2(|\delta(t)|) \\ -c\Gamma(x_t) + 2e^{c\tau}\psi(|\dot{x}(t)|) - 2\psi(|\dot{x}(t-\tau)|). \end{array} \tag{21}$$

From (4) in Assumption 1 and the fact that the function ψ is nondecreasing, we deduce that

$$\begin{array}{c} \dot{U}(t) \leq -\nu(x(t)) + \psi(|\dot{x}(t-\tau)|) + \alpha_2(|\delta(t)|) - c\Gamma(x_t) \\ +2e^{c\tau}\psi\left(\varepsilon|\dot{x}(t-\tau)| + \varpi(x(t)) + \alpha_1(|\delta(t)|)\right) \\ -2\psi(|\dot{x}(t-\tau)|). \end{array} \tag{22}$$

Since the function ψ is nondecreasing, the inequality

$$\begin{array}{c} \psi\left(\varepsilon|\dot{x}(t-\tau)| + \varpi(x(t)) + \alpha_1(|\delta(t)|)\right) \leq \\ \psi\left(2\varepsilon|\dot{x}(t-\tau)|\right) + \psi\left(4\varpi(x(t))\right) + \psi\left(4\alpha_1(|\delta(t)|)\right) \end{array} \tag{23}$$

is satisfied. Consequently,

$$\begin{array}{rcl} \dot{U}(t) & \leq & -\nu(x(t)) + \psi(|\dot{x}(t-\tau)|) + \alpha_2(|\delta(t)|) \\ & & -c\Gamma(x_t) + 2e^{c\tau}\psi\left(2\varepsilon|\dot{x}(t-\tau)|\right) \\ & & +2e^{c\tau}\psi\left(4\varpi(x(t))\right) + 2e^{c\tau}\psi\left(4\alpha_1(|\delta(t)|)\right) \\ & & -2\psi(|\dot{x}(t-\tau)|). \end{array} \tag{24}$$

Now, (7) and (8) in Assumption 2 ensure that $\psi(2\varepsilon|\dot{x}(t-\tau)|) \leq \frac{1}{3}\psi(|\dot{x}(t-\tau)|)$ and $\psi(4\varpi(x(t))) \leq \frac{1}{3}\nu(x(t))$. It follows that

$$\begin{array}{rcl} \dot{U}(t) & \leq & -c\Gamma(x_t) - \nu(x(t)) + \frac{2e^{c\tau}}{3}\psi(|\dot{x}(t-\tau)|) \\ & & +\frac{2e^{c\tau}}{3}\nu(x(t)) - \psi(|\dot{x}(t-\tau)|) \\ & & +\alpha_2(|\delta(t)|) + 2e^{c\tau}\psi\left(4\alpha_1(|\delta(t)|)\right) \\ & \leq & -c\Gamma(x_t) + \left(-1 + \frac{2e^{c\tau}}{3}\right)\nu(x(t)) \\ & & +\left(-1 + \frac{2e^{c\tau}}{3}\right)\psi(|\dot{x}(t-\tau)|) \\ & & +\alpha_2(|\delta(t)|) + 2e^{c\tau}\psi\left(4\alpha_1(|\delta(t)|)\right). \end{array} \tag{25}$$

Since c satisfies (10), the inequality

$$\dot{U}(t) \leq -c\Gamma(x_t) - c\nu(x(t)) - c\psi(|\dot{x}(t-\tau)|) + \alpha_3(|\delta(t)|), \tag{26}$$

with α_3 defined in (14) is satisfied. Consequently, for all $t \in [0, t_\phi)$,

$$\int_0^t \dot{U}(m)dm \leq \int_0^t \alpha_3(|\delta(m)|)dm. \tag{27}$$

By virtue of the continuity of δ and (20), it follows that, if t_ϕ is a finite real number, for all $t \in [0, t_\phi)$,

$$\mu(x(t)) \leq U(x_0) + \int_0^{t_\phi} \alpha_3(|\delta(m)|)dm < +\infty\ . \tag{28}$$

Then, from Assumption 2, we deduce that, for all $t \in [0, t_\phi)$, $|x(t)| \leq \mathcal{B} < +\infty$, with $\mathcal{B} = \kappa_1^{-1}\left(U(x_0) + \int_0^{t_\phi} \alpha_3(|\delta(m)|)dm\right)$. Next, using Assumption 1, we deduce that, almost everywhere over $[0, t_\phi)$,

$$|\dot{x}(t)| \leq \varepsilon|\dot{x}(t-\tau)| + \sup_{|\xi|\leq\mathcal{B}} \varpi(\xi) + \alpha_1\left(\sup_{m\in[0,t_\phi]} |\delta(m)|\right).$$

From this inequality, one can deduce by using the steps method that $\dot{x}$ is essentially bounded over $[0, t_\phi)$. Thus t_ϕ cannot be a real number, i.e. $t_\phi = +\infty$. Hence, all the solutions of (3) are defined over $[-\tau, +\infty)$.

Finally, we observe that (9) in Assumption 2 and (26) give

$$\begin{array}{rcl} \dot{U}(t) & \leq & -c\Gamma(x_t) - c\rho(\mu(x(t))) - c\psi(|\dot{x}(t-\tau)|) \\ & & +\alpha_3(|\delta(t)|), \end{array} \tag{29}$$

which leads to the inequality (12) with $\overline{\rho}$ defined in (13).

III. Backstepping for neutral systems

In this section, we illustrate the usefulness of Theorem 1 by using it to solve a stabilization problem for a particular family of neutral systems in feedback form.

We consider a neutral system of the type

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & H(\dot{x}(t-\tau), x(t), y(t)), \\ \dot{y}(t) & = & u(t), \end{array} \right. \tag{30}$$

where $x \in \mathbb{R}^n$, $\tau > 0$ is a constant, $y \in \mathbb{R}$, H is a locally Lipschitz nonlinear function and $u \in \mathbb{R}$ is the input.

To begin with, we impose an assumption, which is natural in a backstepping context:

Assumption A. *There exists a function $y_s(x)$ of class C^1 such that the system*

$$\dot{x}(t) = H(\dot{x}(t-\tau), x(t), y_s(x(t)) + \delta(t)) \tag{31}$$

satisfies Assumptions 1 and 2 and the function $\frac{\partial y_s}{\partial x}$ is locally Lipschitz.

We also introduce an assumption which is instrumental in our design of stabilizing control laws and has no connection with the classical backstepping approach:

Assumption B. *Let α_1, α_2, ψ be the functions provided by Assumptions 1 and 2. The functions α_1^2 and α_2 are equal to a quadratic function in a neighborhood of the origin. There exist a class $\mathcal{K}_\infty$ function η and two positive real numbers c_p, c_q such that*

$$\psi(m) \geq m\eta(m), \quad \forall m \geq 0 \tag{32}$$

and $\eta(m) = c_p m$ for all $m \in [0, c_q]$. Moreover, the functions α_1, α_2, ψ, η^{-1} are locally Lipschitz.

We are ready to state and prove the following result:

Theorem 2: Assume that the system (30) satisfies Assumptions A and B. Then the control law

$$\begin{array}{c} u(x,y) = -(y - y_s(x)) \\ -\mathrm{sgn}(y - y_s(x)) \left[\alpha_4(|y - y_s(x)|) \right. \\ \left. + \frac{1}{c}\left|\frac{\partial y_s}{\partial x}(x)\right| \eta^{-1}\left(\frac{1}{c}\left|(y - y_s(x))\frac{\partial y_s}{\partial x}(x)\right|\right)\right] \end{array} \tag{33}$$

where $c > 0$ is a constant such that (10) is satisfied and α_4 is the function continuous over $[0, +\infty)$ such that, for all $m > 0$

$$\alpha_4(m) = \frac{\alpha_2(m) + 2e^{c\tau}\psi(4\alpha_1(m))}{m}, \tag{34}$$

globally asymptotically stabilizes the origin of the system (30).

Remark 1a. Assumptions A and B ensure that $u(x, y)$ in (33) is well defined and locally Lipschitz.

Remark 1b. Many extensions of Theorem 2 can be expected. For instance, it can be immediately extended to systems of the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & H(\dot{x}(t-\tau), x(t), y(t)), \\ \dot{y}(t) & = & u(t) + h(x_t, y(t)), \end{array} \right.$$

where h is a continuous functional since the change of feedback $u(t) = -h(x_t, y(t)) + v(t)$ gives a new system of the form (30).

Remark 1c. There is a lot of flexibility in the backstepping design, so that we can modify the control (33). Assumption B can be replaced by functions α_1, α_2, ψ satisfying other types of local homogeneous properties.

Proof. To begin with, we observe that the function α_4 defined in (34) is locally Lipschitz and positive over $(0, +\infty)$ because the functions $\psi(4\alpha_1)$ and α_2 are locally Lipschitz and are equal to positive definite quadratic functions in a neighborhood of the origin.

Next, arguing as we did in the proof of Theorem 1, one can prove that for any initial condition $\phi = (\phi_x, \phi_y) \in C_{\text{in}}$, the system (30) in closed-loop with the feedback (33) admits a unique solution (x, y), which is piecewise continuously differentiable and defined over the interval $[-\tau, t_\phi)$, where t_ϕ is a finite positive real number or $+\infty$. If the maximum of such intervals satisfies $t_\phi < +\infty$, then $(\dot{x}, \dot{y})$ is not essentially bounded over $[-\tau, t_\phi)$.

Next, we construct a Lyapunov functional for (30) in closed-loop with the feedback (33). In the sequel, all equalities and inequalities of $\dot{U}(t)$, $\dot{U}_A(t)$ and $\dot{U}_B(t)$ should be understood to hold *almost everywhere* as long as the solutions are well-defined. The change of variable $z = y - y_s(x)$ transforms the system (30) into

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & H(\dot{x}(t-\tau), x(t), y_s(x(t)) + z(t)), \\ \dot{z}(t) & = & u(x(t), y(t)) - \frac{\partial y_s}{\partial x}(x(t))\dot{x}(t). \end{array} \right. \tag{35}$$

Assumption A ensures that Theorem 1 applies to the x-subsystem of (35). It follows that there are a functional $U : C_{\text{in}} \to [0, +\infty)$ of the type (11) and a constant $c > 0$ satisfying $c \leq 1 - \frac{2e^{c\tau}}{3}$ such that the time derivative of U along the solutions of the x-subsystem of (35) satisfies

$$\dot{U}(t) \leq -\overline{\rho}(U(x_t)) - c\psi(|\dot{x}(t-\tau)|) + \alpha_4(|z(t)|)|z(t)|, \tag{36}$$

where $\overline{\rho}(\ell) = \frac{c}{2}\min\{\ell, \rho(\ell)\}$. It follows that the functional defined by

$$U_A(\phi_x) = U(\phi_x) - c\int_{-\tau}^{0} \psi(|\dot{\phi}_x(\ell)|)d\ell \tag{37}$$

admits a time derivative along the solutions of (35) which satisfies:

$$\dot{U}_A(t) \leq -\overline{\rho}(U(x_t)) - c\psi(|\dot{x}(t)|) + \alpha_4(|z(t)|)|z(t)|. \tag{38}$$

We introduce now the candidate control Lyapunov-Krasovskii functional $U_B : C_{\text{in}} \times \mathbb{R} \to [0, +\infty)$ for the system (35) defined by

$$U_B(\phi_x, z) = U_A(\phi_x) + \frac{1}{2}z^2. \tag{39}$$

Then its time derivative along the solutions of (35) satisfies:

$$\begin{array}{rcl} \dot{U}_B(t) & \leq & -\overline{\rho}(U(x_t)) - c\psi(|\dot{x}(t)|) + \alpha_4(|z(t)|)|z(t)| \\ & & +z(t)\left[u(x(t), y(t)) - \frac{\partial y_s}{\partial x}(x(t))\dot{x}(t)\right]. \end{array} \tag{40}$$

By using (32), we obtain

$$\begin{array}{c} \dot{U}_B(t) \leq -\overline{\rho}(U(x_t)) - c|\dot{x}(t)|\eta(|\dot{x}(t)|) \\ +z(t)u(x(t), y(t)) - z(t)\frac{\partial y_s}{\partial x}(x(t))\dot{x}(t) + \alpha_4(|z(t)|)|z(t)|. \end{array} \tag{41}$$

The property that $ab \leq a\eta(a) + \eta^{-1}(b)b$ holds for $a, b \geq 0$ (which can be proved by considering the cases $b \leq \eta(a)$ and $a \leq \eta^{-1}(b)$ because η is of class $\mathcal{K}_\infty$) ensures that

$$\begin{array}{c} -z(t)\frac{\partial y_s}{\partial x}(x(t))\dot{x}(t) \leq c|\dot{x}(t)|\eta(c|\dot{x}(t)|) \\ + \frac{1}{c}\left|z(t)\frac{\partial y_s}{\partial x}(x(t))\right| \eta^{-1}\left(\frac{1}{c}\left|z(t)\frac{\partial y_s}{\partial x}(x(t))\right|\right). \end{array} \tag{42}$$

It follows that

$$\begin{array}{c} \dot{U}_B(t) \leq -\overline{\rho}(U(x_t)) - c|\dot{x}(t)|\left[\eta(|\dot{x}(t)|) - \eta(c|\dot{x}(t)|)\right] \\ + \frac{1}{c}\left|z(t)\frac{\partial y_s}{\partial x}(x(t))\right| \eta^{-1}\left(\frac{1}{c}\left|z(t)\frac{\partial y_s}{\partial x}(x(t))\right|\right) \\ +z(t)u(x(t), y(t)) + \alpha_4(z(t))|z(t)|. \end{array} \tag{43}$$

Since, for all $z \neq 0$ $u(x,y) = -z - \frac{|z|}{z}\alpha_4(z) - \frac{|z|}{zc}\left|\frac{\partial y_s}{\partial x}(x)\right|\eta^{-1}\left(\frac{1}{c}\left|z\frac{\partial y_s}{\partial x}(x)\right|\right)$, the inequality

$$\begin{aligned} \dot{U}_B(t) \quad &\leq \quad -\overline{\rho}(U(x_t)) - c|\dot{x}(t)|\left[\eta\left(|\dot{x}(t)|\right) - \eta\left(c|\dot{x}(t)|\right)\right] \\ &\quad\quad -z(t)^2 \\ &\leq \quad -\overline{\rho}\left(\tfrac{1}{2}\left(U(x_t) + z(t)^2\right)\right) \\ &\quad\quad -c|\dot{x}(t)|\left[\eta\left(|\dot{x}(t)|\right) - \eta\left(c|\dot{x}(t)|\right)\right] \end{aligned} \tag{44}$$

is satisfied. From $c < 1$ and (11) it follows that $U(\phi_x) \geq U_A(\phi_x) \geq \frac{1}{2}U(\phi_x)$, for all $\phi_x \in C_{\text{in}}$. Therefore

$$U(\phi_x) + z^2 \geq U_B(\phi_x, z) \geq \frac{1}{2}U(\phi_x) + \frac{1}{2}z^2, \tag{45}$$

for all $\phi_x \in C_{\text{in}}$, $z \in \mathbb{R}$. Since the function $\overline{\rho}$ is nondecreasing, it follows that

$$\dot{U}_B(t) \leq -\overline{\rho}\left(\frac{1}{2}U_B(x_t, z(t))\right). \tag{46}$$

Arguing as we did at the end of the proof of Theorem 1, one can prove that the solutions are defined over $[-\tau, +\infty)$. Since $\overline{\rho}$ is of class $\mathcal{K}$, it follows that $t \mapsto U_B(x_t, z(t))$ converges to zero when t goes to the infinity. This fact and the properties of U allow us to conclude.

IV. Applications

In this section, we illustrate Theorem 1 via simple examples.

A. Application to a particular family of one-dimensional systems

We consider the following one-dimensional neutral system

$$\dot{x}(t) = -g(x(t)) + \epsilon\dot{x}(t-\tau) + \delta(t), \tag{47}$$

where $x \in \mathbb{R}$, where g is of class C^1, where ϵ is a real number and where δ is a continuous function which is studied in [15, Chapt. 9] and in [10, Sec. 9.8] in the case where δ is not present. This system is used to modelize a shunted power transmission line. For this system, we establish a stability result via a Lyapunov construction based on Theorem 1. With a view to it, we introduce the function

$$\mu(x) = \int_0^x g(\ell)d\ell \tag{48}$$

and the assumption:

Assumption 3. *The function $x \mapsto xg(x)$ is positive definite and there exists a function ρ of class $\mathcal{K}$ such that, for all $x \in \mathbb{R}$,*

$$\rho(\mu(x)) \leq \frac{1}{2}g(x)^2. \tag{49}$$

We are ready to state and prove the following result:

Corollary 1: Assume that the system (47) satisfies Assumption 3. Then, when

$$|\epsilon| \leq \frac{1}{4\sqrt{6}}, \tag{50}$$

the derivative of the functional $U : C_{\text{in}} \to [0, +\infty)$,

$$U(\phi_x) = \int_0^{\phi_x(0)} g(\ell)d\ell + 2\varepsilon^2 \int_{-\tau}^0 e^{c(\ell+\tau)}\dot{\phi}_x(\ell)^2 d\ell \tag{51}$$

along the solutions of (47) with initial conditions in C_{in} satisfies

$$\dot{U}(t) \leq -\overline{\rho}(U(x_t)) + \alpha_3(|\delta(t)|), \tag{52}$$

for almost all $t \in [0, \infty)$, with $\overline{\rho}(\ell) = \frac{c}{2}\min\{\ell, \rho(\ell)\}$, where ρ is the function in (49) and $\alpha_3(\ell) = \left(32e^{c\tau}\epsilon^2 + 1\right)\ell^2$, where c is any real number such that $0 < c \leq 1 - \frac{2e^{c\tau}}{3}$.

Remark 2. The functional U defined in (51) is nonnegative because the function $x \mapsto xg(x)$ is positive definite.

Remark 3. One can prove that Assumption 3 is satisfied if and only if the function $x \mapsto g(x)^2$ is positive definite and $\inf\limits_{|x|\geq 1}\{g(x)^2\} > 0$. Thus, Corollary 1 does not need the requirement that $x \mapsto g(x)^2$ is radially unbounded, which is imposed in [10, Sec. 9.8].

Proof of Corollary 1. Let us check now that, under the conditions of Corollary 1, the assumptions of Theorem 1 are satisfied.

Since the function $x \mapsto xg(x)$ is positive definite, it follows that the function μ is positive definite and $\lim_{x\to+\infty}\mu(x)$ and $\lim_{x\to-\infty}\mu(x)$ are positive real numbers or $+\infty$. Since ρ is of class $\mathcal{K}$, (49) implies that there exists a constant $g_m > 0$ such that $|g(x)| \geq g_m$ for all $x \in (-\infty, -1] \cup [1, +\infty)$. It follows that μ is radially unbounded. Therefore there are two functions κ_1 and κ_2 of class $\mathcal{K}_\infty$ such that, for all $x \in \mathbb{R}^n$, $\kappa_1(|x|) \leq \mu(x) \leq \kappa_2(|x|)$.

Next, the right hand side of (47) satisfies Assumption 1 with $\varpi(m) = |g(m)|$, $\varepsilon = |\epsilon|$ and $\alpha_1(m) = m$ for all $m \geq 0$. We observe that, for all $(a, b, \Delta) \in \mathbb{R}^3$,

$$\mu'(b)[-g(b) + \epsilon a + \Delta] = -g(b)^2 + \epsilon g(b)a + g(b)\Delta. \tag{53}$$

We deduce from Young's inequality that

$$\mu'(b)[-g(b) + \epsilon a + \Delta] \leq -\nu(b) + \psi(|a|) + \Delta^2, \tag{54}$$

with $\nu(x) = \frac{1}{2}g(x)^2$, $\psi(m) = \varepsilon^2 m^2$ and $\alpha_2(m) = m^2$. Moreover, we have established that there are two functions κ_1 and κ_2 such that the inequalities (5) are satisfied. Next, for all $\ell \geq 0$,

$$3\psi(2\varepsilon\ell) = 3\varepsilon^2(2\varepsilon\ell)^2 = 12\varepsilon^4\ell^2 \leq 12\varepsilon^2\psi(\ell). \tag{55}$$

From (50), it follows that, for all $\ell \geq 0$,

$$3\psi(2\varepsilon\ell) \leq \psi(\ell). \tag{56}$$

Moreover, from (50), it follows that, for all $x \in \mathbb{R}$,

$$3\psi(4|g(x)|) = 48\varepsilon^2|g(x)|^2 \leq \frac{1}{2}g(x)^2 = \nu(x). \tag{57}$$

It follows that Assumption 2 is satisfied. Hence, Theorem 1 applies. This allows us to conclude.

B. Examples

For the important family of neutral systems Corollary 1 focuses on, this section presents an example which illustrate the robustness of the types (16) and (15), respectively. The second result illustrates Theorem 2.

First example. Consider the one-dimensional system

$$\dot{x}(t) = -x(t)^3 + \epsilon\dot{x}(t-\tau) + \delta(t). \tag{58}$$

Even when $\epsilon = 0$ and δ is not present, the origin of this system is not locally exponentially stable. Moreover, the function $g(x) = x^3$ is not globally Lipschitz.

The functions $\mu(x) = \frac{1}{4}x^4$, $g(x)^2 = x^6$ satisfy $\rho(\mu(x)) \leq \frac{1}{2}g(x)^2$, with $\rho(\ell) = \frac{\ell^2}{1+\ell}$. Since this function is of class $\mathcal{K}$, Assumption 3 is satisfied. Thus, Corollary 1 provides with the functional

$$U(\phi_x) = \frac{1}{4}\phi_x(0)^4 + 2\epsilon^2 \int_{-\tau}^{0} e^{c(\ell+\tau)}\dot{\phi}_x(\ell)^2 d\ell,$$

where c is any real number such that $0 < c \leq 1 - \frac{2e^{c\tau}}{3}$. Its derivative along the trajectories of (58) satisfies

$$\dot{U}(t) \leq -\overline{\rho}(U(x_t)) + \alpha_3(|\delta(t)|),$$

with $\overline{\rho}(\ell) = \frac{c}{2}\frac{\ell^2}{1+\ell}$, and $\alpha_3(\ell) = \left(32e^{c\tau}\epsilon^2 + 1\right)\ell^2$. Since the function $\overline{\rho}$ is of class $\mathcal{K}_\infty$, we deduce that the solutions of (58) satisfy an inequality of the type (16).

Second example. Consider the two-dimensional system

$$\begin{array}{rcl} \dot{x}(t) & = & \epsilon\dot{x}(t-\tau) + y(t), \\ \dot{y}(t) & = & u(t), \end{array} \tag{59}$$

with $x(t), y(t), u(t), \epsilon \in \mathbb{R}$. Let $y_s(x) = -\frac{x}{\sqrt{1+x^2}}$ for all $x \in \mathbb{R}$. Then Assumption 1 is satisfied by the system

$$\dot{x}(t) \quad = \quad \epsilon\dot{x}(t-\tau) + y_s(x(t)), \tag{60}$$

with $\varpi(l) = \frac{|l|}{\sqrt{1+l^2}}$, $\alpha_1(l) = l$, $\varepsilon = |\epsilon|$. Moreover, Assumption 2 is satisfied by the system (60) with $\mu(x) = \displaystyle\int_0^x \frac{m}{\sqrt{1+m^2}}dm$, $\nu(x) = \frac{x^2}{2(1+x^2)}$, $\psi(m) = \varepsilon^2 m^2$, $\alpha_2(m) = m^2$, provided $\varepsilon \leq \frac{1}{4\sqrt{6}}$. Thus the system (59) satisfies Assumptions A and B with $\eta(m) = \varepsilon^2 m$. From Theorem 2, we deduce that if $\varepsilon = \frac{1}{4\sqrt{6}}$ the origin of the system (59) is globally asymptotically stabilized by the the control law

$$u(x,y) = -\left(2 + \tfrac{e^{c\tau}}{3}\right)\left(y + \tfrac{x}{\sqrt{1+x^2}}\right) - \frac{96\left(y + \frac{x}{\sqrt{1+x^2}}\right)}{c^2(1+x^2)^3} \tag{61}$$

with $c = \min\left\{\frac{1}{6}, \frac{1}{\tau}\ln\left(\frac{5}{4}\right)\right\}$. The above choice of $y_s(x)$ allows $u(x,y)$ to be bounded in terms of x.

V. Conclusion

We have proposed a new stability analysis technique for neutral systems based on the construction of an adequate Lyapunov-Krasovskii functional. The technique applies to broad families of neutral systems which include systems which are not locally exponentially stable and not globally Lipschitz and makes it possible to construct stabilizing control laws for systems in feedback form under appropriate assumptions. Much remains to be done. In particular, extensions to time-varying systems, to systems with several delays are expected, as long as applications to other control problems for neutral systems than the one solved in Section III.

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