

# Global Output Feedback Stabilization of a Chemostat With an Arbitrary Number of Species

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## Abstract

This paper studies the global output feedback stabilization problem for a chemostat with an arbitrary number of species competing for a single nutrient source. It is shown that coexistence can be achieved by means of a feedback control of the substrate input concentration and an appropriate choice of a constant dilution rate. The input substrate feedback controller proposed is a dynamic time-varying output feedback, with the substrate concentration considered as the output. The result relies on the construction of a novel nonlinear observer and is established via a Lyapunov-based proof.

**Keywords.** Chemostats, output feedback, global stabilization.

## I. INTRODUCTION

The basic model for a chemostat with  $N$  competing species and one nutrient source, and all the yield coefficients chosen equal to 1, is

$$\begin{cases} \dot{s} &= D[s_{in} - s] - \sum_{j=1}^N \mu_j(s)x_j, \\ \dot{x}_i &= [\mu_i(s) - D]x_i, \quad i = 1, \dots, N, \end{cases} \quad (1)$$

evolving on  $(0, \infty)^{N+1}$ , where  $s$  is the substrate concentration;  $D$  is the flow rate;  $s_{in}$  is the input nutrient concentration;  $\mu_i$  give the per capita growth rate;  $x_i$  is the concentration of the  $i$ th population. The growth functions  $\mu_i$  are increasing on  $[0, +\infty)$ ,  $D$  and  $s_{in}$  are nonnegative.

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This celebrated model is presented and studied in [14] and has been studied extensively for decades because this system may be used to model the competition of several organisms for a single nutrient source. The competitive exclusion principle states that if  $D$  and  $s_{in}$  are constants, then, generically, at most one species survives. This is at odds with the observation that in real ecological systems, it is common for many species to coexist in equilibrium on one nutrient source. This paradox has motivated a great deal of research: many contributions are devoted to modifying the model (1) to ensure coexistence of the species. Generally speaking, the past literature devoted to this problem can be classified into two categories. The first category of papers is devoted to the analysis of specific chemostats and studies conditions for coexistence of the species. Several researchers have shown that in some special cases, the equations (1) do not correctly describe a bioreactor. For instance in [10], it is shown that in some situations the yield coefficients are not constants but instead are functions of  $s$  and then coexistence through limit cycles may occur. In the papers [4], [5] it is shown that models taking into account intra-specific competition may admit a positive globally asymptotically stable equilibrium point. In [11], existence of a time-varying dilution rate resulting in a system that admits solutions for which the coexistence of an arbitrary number of species occurs is established. The other category of papers is devoted to the control problem consisting in determining feedback laws which ensure persistence, when  $D$  and  $s_{in}$  are regarded as inputs which can be chosen by an operator. In [1] and in [3], stabilizing feedback laws depending only on a sum of the species concentrations are used to stabilize a chemostat with two species. The paper [9] is concerned with the problem of stabilizing a periodic trajectory of the system (1) when there are two species.

A common feature of the papers devoted to the problem of ensuring coexistence through feedback control before the recent papers [7], [8] is that they ensure coexistence of two and only two species. In order to show that in some cases robust coexistence may be ensured by feedback control, we designed in [7], [8] state feedback laws which globally asymptotically stabilize a positive periodic trajectory of (1) and thereby ensure the persistence in a chemostat of an arbitrary number of species independently from the values of the initial concentrations. The control laws proposed are positive everywhere. The dilution rate  $D$  is chosen equal to a specific constant and  $s_{in}$  is taken equal to a function of class  $C^1$  which depends on the concentration of each species but is independent of the substrate concentration. With the substrate concentration as the only measured variable, we also constructed a linear periodic dynamic output feedback

which locally exponentially stabilizes a periodic trajectory when the output is the substrate concentration i.e.  $y = s$ .

In the present paper, we extend the results of [7], [8] and [9] by designing a dilution rate and input substrate feedback controllers when only the substrate concentration  $s$  is measured. More precisely, we will achieve the coexistence by designing a novel output-feedback controller that globally asymptotically stabilizes a periodic reference state trajectory of the system (1), instead of local asymptotic trajectory-stabilization as accomplished in [7], [8]. It is worth mentioning that, in practice, measuring the values of the concentration of each species is not feasible but measuring the substrate concentration is. Therefore considering  $s$  as the output is a reasonable choice which is made for instance in [12]. The dynamic output feedback we shall propose relies on an observer. For technical reasons that will appear along our constructions, its analytic expression is slightly different from the one of the state feedback proposed in [7], [8]. Our stability analysis is based on the explicit construction of a Lyapunov function and the LaSalle Invariance Principle. Our paper is organized as follows. The main result is stated in Section II. Its proof is in Section III. An illustrative example is given in Section IV. Concluding remarks are drawn in Section V.

*Throughout this paper:* • We will say that a point is positive if all its components are positive. • We call reference state trajectory for the system (1) any bounded function  $(s_r(t), x_{1r}(t), \dots, x_{Nr}(t))$  such that (i) There exists  $\varepsilon_1 > 0$  such that for all  $t \geq 0$ ,  $s_r(t) \geq \varepsilon_1$ ,  $x_{ir}(t) \geq \varepsilon_1$  for all  $i \in \{1, \dots, N\}$ . (ii) There exist  $D_r(t)$ ,  $s_{inr}(t)$  and two constants  $\varepsilon_2 > 0$ ,  $\varepsilon_3 > 0$  such that for all  $t \geq 0$ ,  $\varepsilon_3 \geq D_r(t) \geq \varepsilon_2$ ,  $\varepsilon_3 \geq s_{inr}(t) \geq \varepsilon_2$  and

$$\begin{cases} \dot{s}_r(t) &= D_r(t)[s_{inr}(t) - s_r(t)] - \sum_{j=1}^N \mu_j(s_r(t))x_{jr}(t), \\ \dot{x}_{ir}(t) &= [\mu_i(s_r(t)) - D_r(t)]x_{ir}(t), \quad i = 1, \dots, N. \end{cases} \quad (2)$$

• We will say that a trajectory of the system (1) is globally asymptotically stable if all the solutions of (1) with a *positive* initial condition converge to this trajectory. • The arguments of the functions will be omitted or simplified whenever no confusion can arise from the context.

## II. ASSUMPTIONS AND STABILIZATION THEOREM

In this section, we give the main result of our work under assumptions similar to those introduced in [7], [8].

### A. Assumptions

**Assumption H1.** The functions  $\mu_i$  are of class  $C^2$ , such that  $\mu_i(0) = 0$  and  $\mu'_i(\ell) > 0$  for all  $\ell \in \mathbb{R}$  and there exists  $\mu_B > 0$  such that

$$\sup_{\{\ell \in \mathbb{R}\}} \mu'_i(\ell) \leq \mu_B . \quad (3)$$

**Assumption H2.** There exist a positive function  $p(t)$  of class  $C^1$ , periodic of period  $T > 0$  and a positive real number  $D_*$  such that, for all  $i \in \{1, \dots, N\}$ ,

$$D_* = \frac{1}{T} \int_0^T \mu_i(p(\ell)) d\ell . \quad (4)$$

**Assumption H3.** If the real numbers  $c_j$ ,  $j = 1$  to  $N$ , are such that, for all  $r \in [m_1, m_2]$ , with  $m_1 = \min_{t \in [0, T]} p(t)$  and  $m_2 = \max_{t \in [0, T]} \max\{p(t), -\dot{p}(t)\}$ , the equality  $\sum_{j=1}^N c_j \mu'_j(r) = 0$  is satisfied then,  $c_j = 0$  for all  $j \in \{1, \dots, N\}$ .

**Assumption H4.** There exists  $c \in [m_1, m_2]$  such that, for all  $i \neq j$ ,  $\mu_i(c) \neq \mu_j(c)$ .

### B. Discussion of the assumptions

Assumptions H1 to H4 are introduced and explained in [7], [8]. However, we recall some important facts about them.

- Assumption H1 is satisfied by most of the growth functions classically used in the literature. As we shall see later, the parameter  $\mu_B$  is present implicitly in the expressions of the control laws we derive and, in general, the largest it is, the slowest is the speed of convergence of our closed-loop system. However, this drawback does not significantly limit the interest of our control design because our main control objective is more to ensure persistence than track a specific trajectory. Moreover the basin of attraction we obtain is the entire positive orthant and so it does not depend on  $\mu_B$ .
- Assumption H2 is crucial: the system (1) admits a periodic positive reference state trajectory if and only if it is satisfied. It is a restrictive assumption in the sense that if one picks an arbitrary microbial ecosystem in the nature, then, in general, the family of all the species typically contains a huge number of elements and is not such that Assumption H2 is satisfied. However our result using Assumption H2 is of interest because (i) in artificial ecosystems, specific microorganisms

such that Assumption H2 is satisfied can be selected and then coexistence of species, which is a phenomenon frequently observed in nature, can be experimentally reproduced; (ii) this assumption, in combination with Assumption H3, helps to understand how stable oscillatory behaviors can be modeled; (iii) our results complement the important result of [11] because Assumption H2 is equivalent to the necessary and sufficient conditions introduced in [11] to ensure that an arbitrary number of coexisting periodically varying species can be produced using a periodic input concentration and a fixed dilution rate.

- Assumption H3 provides an important information about the stabilizability of a trajectory of a chemostat with more than two species. In contrast with the case  $N = 2$ , we have shown in [7], [8] that, when  $N > 2$ , the periodic positive reference state trajectory deduced from Assumption H2 is not necessarily stabilizable if Assumption H3 is violated. Determining the key role played by Assumption H3 enabled us in [7], [8] to solve for the first time the problem of ensuring through state feedback the global and stable coexistence of more than two species of microorganisms in a chemostat.
- Assumption H4 ensures that the observer we will construct in Section III asymptotically converges to the solutions of (1) for some appropriate choices of the control law. This assumption is not restrictive.
- It is worth mentioning that in [7, Section 6] and [8, Section 6] it is shown that Assumptions H1 to H4 are satisfied by a set of functions including classical and important families of growth functions, and in particular families of Monod functions.

### C. Main result

Let us introduce the function

$$p_\delta(t) = p(\delta t) \tag{5}$$

where  $\delta$  is a positive parameter. Our main result is as follows:

*Theorem 1:* Consider the system (1) endowed with the output  $y = s$ . Assume that Assumptions H1 to H4 are satisfied and let the dilution rate be equal to the constant  $D_*$ . Then, for any constant  $\delta$  sufficiently small, one can construct a dynamic extension

$$\dot{h} = F(t, y, h) \tag{6}$$

with  $h \in \mathbb{R}^{N+1}$ , where  $F : \mathbb{R}^{N+3} \rightarrow \mathbb{R}^{N+1}$  is of class  $C^1$  and periodic with respect to its first argument, and find a control law  $s_{in} : \mathbb{R}^{N+3} \rightarrow \mathbb{R}$ , of class  $C^1$ , positive everywhere and periodic such that the system (1) in closed-loop with  $s_{in}(t, s, h)$  admits the positive periodic trajectory

$$(s_r(t), x_{1r}(t), \dots, x_{Nr}(t)) = (p_\delta(t), e^{\xi_{1r}(t)}, \dots, e^{\xi_{Nr}(t)}) \quad (7)$$

with

$$\xi_{ir}(t) = \frac{\delta}{T} \int_{t-\frac{T}{\delta}}^t \int_{\ell}^t [\mu_i(p_\delta(w)) - D_*] dw d\ell + g_i, \quad (8)$$

where the  $g_i$ 's are arbitrary numbers, as a globally asymptotically stable reference state trajectory.

**Remark 1.** The family of stabilizable reference trajectories in (7), (8) and the one considered in [7] and [8] are the same.

### III. PROOF OF THEOREM 1

The proof splits up into five parts. First, we give technical results which will be instrumental in establishing the main result. Second, we introduce a general family of observers. Third, to ease the designs of the observer and of the control laws, we perform changes of coordinates and feedback. Fourth, a stabilizing control law is determined. Finally, in the last part of the proof, we establish that some of the control laws we have selected are positive and we demonstrate the desired stability property for the closed-loop system.

#### A. Preliminary results

The first technical lemma is classical, a proof of which can be found in [14]. The second and the third lemmas can be directly checked.

*Lemma 1:* Consider the system (1) under the assumptions of Theorem 1. Moreover assume that  $D$  is constant and that  $s_{in} = q(t) + f(t, s)$  where  $q(t)$  is a positive function and  $f$  is a function of class  $C^1$  such that, for all  $t \geq 0$ ,  $f(t, 0) = 0$ . If  $s(0) \in (0, +\infty)$  and, for  $i = 1$  to  $N$ ,  $x_i(0) \in (0, +\infty)$ , then, for all  $t \geq 0$ ,  $s(t) \in (0, +\infty)$  and, for  $i = 1$  to  $N$ ,  $x_i(t) \in (0, +\infty)$ .

*Lemma 2:* Consider the system (1) under the assumptions of Theorem 1. Then the changes of variables  $\xi_i = \ln(x_i)$  transform the system (1) into

$$\begin{cases} \dot{s} &= D[s_{in} - s] - \sum_{j=1}^N \mu_j(s) e^{\xi_j}, \\ \dot{\xi}_i &= [\mu_i(s) - D], \quad i = 1, \dots, N. \end{cases} \quad (9)$$

*Lemma 3:* Assume that Assumptions H3 and H4 are satisfied with a function  $p$ . Then, for any positive real number  $\delta$ , Assumptions H3 and H4 are satisfied with the corresponding function  $p_\delta$  defined in (5). Then  $p_\delta$  is such that Assumptions H3 and H4 are satisfied. The functions  $\xi_{ir}$  defined in (8) are periodic of period  $\frac{T}{\delta}$  and satisfy, for all  $t \in \mathbb{R}$ ,

$$\dot{\xi}_{ir}(t) = \mu_i(p_\delta(t)) - D_* . \quad (10)$$

### B. Nonlinear candidate observer

In this section, we introduce a family of candidate observers for the system (9):

$$\begin{cases} \dot{\hat{s}} &= D[s_{in} - s] - \sum_{j=1}^N \mu_j(s) e^{\hat{\xi}_j} + \alpha(t, \hat{s} - s, h) , \\ \dot{\hat{\xi}}_i &= [\mu_i(s) - D] + \bar{s} \mu_i(s) e^{\hat{\xi}_i} , \quad i = 1, \dots, N , \end{cases} \quad (11)$$

with  $\hat{\xi} = (\hat{\xi}_1, \dots, \hat{\xi}_N)$ ,  $h = (\hat{s}, \hat{\xi})$  and where the functions  $\alpha$  and the  $\beta_i$ 's are to be selected. To determine systems of the form (11) whose solutions converge to the solutions of the system (1), we introduce the error variables

$$\bar{s} = \hat{s} - s , \quad \bar{\xi}_j = \hat{\xi}_j - \xi_j , \quad \bar{\xi} = (\bar{\xi}_1, \dots, \bar{\xi}_N) . \quad (12)$$

and write the error equation

$$\begin{cases} \dot{\bar{s}} &= - \sum_{j=1}^N \mu_j(s) e^{\hat{\xi}_j} [1 - e^{-\bar{\xi}_j}] + \alpha(t, \bar{s}, h) , \\ \dot{\bar{\xi}}_i &= \bar{s} \mu_i(s) e^{\hat{\xi}_i} , \quad i = 1, \dots, N . \end{cases} \quad (13)$$

We introduce the positive definite function

$$U(\bar{s}, \bar{\xi}) = \frac{1}{2} \bar{s}^2 + \sum_{i=1}^N \left( e^{-\bar{\xi}_i} - 1 + \bar{\xi}_i \right) . \quad (14)$$

Next, we consider solutions of (1) and (11) with positive initial conditions  $s(0)$ ,  $\hat{s}(0)$ . Then, according to Lemma 1,  $s(t) > 0$  for all  $t \geq 0$  and the derivative of  $U$  along the trajectories of (13) satisfies

$$\dot{U} = - \sum_{i=1}^N \bar{s} \mu_i(s) e^{\hat{\xi}_i} \left( 1 - e^{-\bar{\xi}_i} \right) + \bar{s} \alpha(t, \bar{s}, h) + \sum_{i=1}^N \left( -e^{-\bar{\xi}_i} + 1 \right) \bar{s} \mu_i(s) e^{\hat{\xi}_i} \quad (15)$$

which simplifies as

$$\dot{U} = \bar{s} \alpha(t, \bar{s}, h) . \quad (16)$$

This equality implies that  $\alpha$  can be designed so that the corresponding system (13) is stable: for instance the choice  $\alpha(t, \bar{s}, h) = -\bar{s}$  renders the right hand side of (16) nonpositive and therefore results in a stable system (13). However, this property will not imply that the solutions of the system (13) converge to the origin: only the first component of the solutions is guaranteed to converge to zero. In fact, at that point, we do not know if  $\alpha$  can be chosen in such a way that (11) becomes an observer for (1). Fortunately, we will see that this can be done when a specific control law is chosen for  $s_{in}$ .

### C. Transformations of the observer

The system (11) with  $D = D_*$  and the error equation yield the equations

$$\begin{cases} \dot{\hat{s}} &= D_*[s_{in} - \hat{s} + \bar{s}] - \sum_{j=1}^N \mu_j(s) e^{\hat{\xi}_j} + \alpha(t, \bar{s}, h), \\ \dot{\hat{\xi}}_i &= [\mu_i(\hat{s}) - D_*] + \bar{s} \mu_i(s) e^{\hat{\xi}_i} + [\mu_i(s) - \mu_i(\hat{s})], \quad i = 1, \dots, N, \\ \dot{\bar{s}} &= \alpha(t, \bar{s}, h) - \sum_{j=1}^N \mu_j(s) e^{\hat{\xi}_j} [1 - e^{-\bar{\xi}_j}], \\ \dot{\bar{\xi}}_i &= \mu_i(s) e^{\hat{\xi}_i} \bar{s}, \quad i = 1, \dots, N. \end{cases} \quad (17)$$

These equations are fundamental: if, for any  $\delta > 0$  smaller than a fixed constant, we can find a control law  $s_{in}$  and a function  $\alpha$  so that all the trajectories of (17), with any initial condition  $(\hat{s}(0), \hat{\xi}(0), \bar{s}(0), \bar{\xi}(0))$  such that  $\hat{s}(0) > 0$ ,  $\hat{s}(0) > \bar{s}(0)$ , converge to the periodic trajectory  $(s_r(t), \xi_r(t), 0, 0)$  with  $s_r, \xi_r$  defined in (7) and (8) then Theorem 1 is established.

To simplify the remaining part of the proof, we perform the change of feedback

$$s_{in} = \frac{1}{D_*} \left[ u + D_* p_\delta(t) + \dot{p}_\delta(t) + \sum_{j=1}^N \mu_j(s) e^{\hat{\xi}_j} - D_* \bar{s} - \alpha(t, \bar{s}, h) \right] \quad (18)$$

where  $u$  is the new input and introduce the notation

$$\gamma_i(\bar{s}, \hat{s}, \hat{\xi}_i) = \bar{s} \mu_i(\hat{s} - \bar{s}) e^{\hat{\xi}_i} + \mu_i(\hat{s} - \bar{s}) - \mu_i(\hat{s}). \quad (19)$$

These modifications transform the  $(\hat{s}, \hat{\xi})$ -subsystem of (17) into

$$\begin{cases} \dot{\hat{s}} &= -D_* \hat{s} + D_* p_\delta(t) + \dot{p}_\delta(t) + u, \\ \dot{\hat{\xi}}_i &= \mu_i(\hat{s}) - D_* + \gamma_i(\bar{s}, \hat{s}, \hat{\xi}_i), \quad i = 1, \dots, N. \end{cases} \quad (20)$$

Next, we perform the time-varying change of coordinates already used in [7], [8]:

$$a = \hat{s} - p_\delta(t), \quad (21)$$



$$b_j = \hat{\xi}_j - \xi_{jr}(t), \quad b = (b_1, \dots, b_N). \quad (22)$$

Using the equality  $s = \hat{s} - \bar{s}$  and (8), the system (17) can be rewritten as

$$\begin{cases} \dot{a} &= -D_* a + u, \\ \dot{b}_i &= [\mu_i(a + p_\delta(t)) - \mu_i(p_\delta(t))] + \theta_i(t, \bar{s}, a, b_i), \quad i = 1, \dots, N, \\ \dot{\bar{s}} &= \alpha(t, \bar{s}, h) - \sum_{j=1}^N \mu_j(\hat{s} - \bar{s}) e^{\hat{\xi}_j} [1 - e^{-\bar{\xi}_j}], \\ \dot{\bar{\xi}}_i &= \mu_i(\hat{s} - \bar{s}) e^{\hat{\xi}_i} \bar{s}, \quad i = 1, \dots, N, \end{cases} \quad (23)$$

with

$$\theta_i(t, \bar{s}, a, b_i) = \gamma_i(\bar{s}, a + p_\delta(t), b_i + \xi_{ir}(t)). \quad (24)$$

#### D. Global output feedback stabilization

Our next objective is to determine, by a Lyapunov approach, functions  $\alpha$  and  $u$  that lead to the asymptotic stability of the system (23) at the origin.

Consider the positive definite function

$$V(a, b) = \frac{1}{2} a^2 + \epsilon [R(b) - 1], \quad (25)$$

where  $\epsilon$  is a positive real number to be chosen later and

$$R(b) = \sqrt{\sum_{j=1}^N b_j^2 + 1}. \quad (26)$$

Its derivative along the trajectories of (23) is

$$\dot{V} = -D_* a^2 + au + \frac{\epsilon}{R(b)} \sum_{j=1}^N b_j [\mu_j(a + p_\delta(t)) - \mu_j(p_\delta(t)) + \theta_j(t, \bar{s}, a, b_j)]. \quad (27)$$

Since the function  $R$  is positive everywhere, we can take

$$u(t, a, b) = -\frac{\epsilon}{R(b)} \sum_{j=1}^N b_j \int_0^1 \mu'_j(p_\delta(t) + \ell a) d\ell. \quad (28)$$

Noticing that, for such a choice, the equality

$$au(t, a, b) = -\frac{\epsilon}{R(b)} \sum_{j=1}^N b_j [\mu_j(a + p_\delta(t)) - \mu_j(p_\delta(t))] \quad (29)$$

holds everywhere, we deduce that our control law gives

$$\dot{V} = -D_* a^2 + \frac{\epsilon}{R(b)} \sum_{j=1}^N b_j \theta_j(t, \bar{s}, a, b_j). \quad (30)$$

Since the right hand side of (30) may take positive values, we cannot deduce that the  $(a, b)$ -subsystem of (23) is stable. However, (30) in combination with (16), allows us to find a candidate Lyapunov function whose derivative along the trajectories of the system (23) is non-positive when is made an appropriate choice for the function  $\alpha$ . This function is simply

$$W(a, b, \bar{s}, \bar{\xi}) = V(a, b) + KU(\bar{s}, \bar{\xi}) \quad (31)$$

where  $K$  is a positive real number that we introduce to give more flexibility to our control design, which is not ended yet because  $\alpha$  still has to be chosen. From (16) and (30), we deduce that its derivative along the trajectories of the system (23) satisfies

$$\dot{W} = -D_* a^2 + \frac{\epsilon}{R(b)} \sum_{j=1}^N b_j \theta_j(t, \bar{s}, a, b_j) + K \bar{s} \alpha(t, \bar{s}, h) . \quad (32)$$

Bearing in mind the expression of the  $\theta_j$ 's, we choose

$$\alpha(t, \bar{s}, h) = -D_* \bar{s} - \frac{\epsilon}{KR(b)} \sum_{j=1}^N b_j \left[ \mu_j(\hat{s} - \bar{s}) e^{\hat{\xi}_j} - \int_0^1 \mu'_j(\hat{s} - \ell \bar{s}) d\ell \right] \quad (33)$$

and obtain the equality

$$\dot{W} = -D_* a^2 - KD_* \bar{s}^2 \leq 0 \quad (34)$$

and the system (23) becomes

$$\left\{ \begin{array}{l} \dot{a} = -D_* a - \frac{\epsilon}{R(b)} \sum_{j=1}^N b_j \int_0^1 \mu'_j(p_\delta(t) + \ell a) d\ell, \\ \dot{b}_i = \mu_i(a + p_\delta(t)) - \mu_i(p_\delta(t)) + \theta_i(t, \bar{s}, \hat{s}, b_i), \quad i = 1, \dots, N, \\ \dot{\bar{s}} = -D_* \bar{s} - \frac{\epsilon}{KR(b)} \sum_{j=1}^N b_j \left[ \mu_j(\hat{s} - \bar{s}) e^{\hat{\xi}_j} - \int_0^1 \mu'_j(\hat{s} - \ell \bar{s}) d\ell \right] \\ \quad - \sum_{j=1}^N \mu_j(\hat{s} - \bar{s}) e^{\hat{\xi}_j} [1 - e^{-\bar{\xi}_j}], \\ \dot{\bar{\xi}}_i = \mu_i(\hat{s} - \bar{s}) e^{\hat{\xi}_i} \bar{s}, \quad i = 1, \dots, N. \end{array} \right. \quad (35)$$

### E. Stability analysis

The right hand side of (34) is nonpositive but not negative definite. Therefore the system (35) is stable but not necessarily globally asymptotically stable. To establish this result, we need to apply the LaSalle Invariance Principle. Before doing this, we need first to establish that, for some values of  $\delta$ ,  $\epsilon$ ,  $K$ , the control  $s_{in}$  we have constructed is positive everywhere in our domain of interest because if this sign constraint is violated, then, in practice, this control cannot be used.

1) *Sign of the control law:* We prove in the appendix the following result:

*Lemma 4:* When the positive constants  $\delta$ ,  $\epsilon$ ,  $K$  are such that

$$m_1 - \frac{\delta m_2}{D_*} - \left(1 + \frac{1}{K}\right) \frac{\epsilon \sqrt{N} \mu_B}{D_*} > 0, \quad K \geq \epsilon \quad (36)$$

where  $m_1$  and  $m_2$  are the constants defined in Assumption H3, the function  $s_{in}(t, s, h)$  defined in (18), (28), and whose analytic expression is

$$s_{in}(t, s, h) = p_\delta(t) + \frac{\dot{p}_\delta(t)}{D_*} - \frac{\epsilon}{D_* R(b)} \sum_{j=1}^N b_j \int_0^1 \mu'_j((1-\ell)p_\delta(t) + \ell \hat{s}) d\ell \\ + \frac{1}{D_*} \left[ \sum_{j=1}^N \mu_j(s) e^{\hat{\xi}_j} + \frac{\epsilon}{K R(b)} \sum_{j=1}^N b_j \left( \mu_j(s) e^{\hat{\xi}_j} - \int_0^1 \mu'_j((1-\ell)\hat{s} + \ell s) d\ell \right) \right], \quad (37)$$

is larger, for all  $t \in \mathbb{R}$ ,  $h \in \mathbb{R}^{N+1}$ , and  $s \in [0, +\infty)$ , than a positive constant.

**Remark 2.** Two questions about the inequality (36) arise. First, it is possible to find parameters  $\delta$ ,  $\epsilon$ ,  $K$  such that this inequality is satisfied? One can easily give a positive answer by noticing that (36) is satisfied in the particular case where  $\delta \leq \frac{m_1 D_*}{4m_2}$ ,  $\epsilon \leq \min \left\{ 1, \frac{1}{8N} \left( \frac{m_1 D_*}{\mu_B} \right)^2 \right\}$ ,  $K = \sqrt{\epsilon}$ . Second, is it possible to relax the requirement (36)? Clearly, there is no reason to believe that the answer is no. On the other hand, one cannot expect to perform the design without having constraints on the parameters  $\delta$ ,  $\epsilon$ ,  $K$ . In particular, one cannot choose arbitrarily large values for  $\delta$ . Indeed, our definition of reference trajectory implies that, if  $D_r(t) = D_*$  and  $s_r(t) = p(\delta t)$  (see (2), (7) and (8)) then

$$\delta \dot{p}(\delta t) + D_* p(\delta t) + \sum_{j=1}^N \mu_j(p(\delta t)) e^{\frac{\delta}{T} \int_{t-\frac{T}{\delta}}^t \int_\ell^t [\mu_j(p(\delta w)) - D_*] dw d\ell + g_j} > 0 \quad (38)$$

for all  $t \geq 0$  where the  $g_i$ 's are arbitrary numbers. Since, on the one hand, one can show through lengthy but simple calculations that the function  $D_* p(\delta t) + \sum_{j=1}^N \mu_j(p(\delta t)) e^{\frac{\delta}{T} \int_{t-\frac{T}{\delta}}^t \int_\ell^t [\mu_j(p(\delta w)) - D_*] dw d\ell + g_j}$  is positive and smaller than a constant independent from  $\delta$  and on the other hand, necessarily  $\lim_{\delta \rightarrow +\infty} (\delta \inf_{t \geq 0} \{\dot{p}(\delta t)\}) = -\infty$ , we deduce that, to guarantee that (38) is satisfied,  $\delta$  must be smaller than a certain constant. A similar reasoning cannot be applied to  $\epsilon$  and  $K$ : however, the intuition suggests that, due to the sign constraint of  $s_{in}$ , one cannot have an arbitrarily fast convergence and this limitation is reflected, in the control law we propose, by the fact that arbitrary constants  $\epsilon$  and  $K$  are not allowed. However, one cannot claim that the control laws we obtain give to the trajectories the fastest possible speed of convergence and the simulations in Section IV reveal a fast convergence of the observer and a slow convergence of the solutions to the reference trajectory. However, the main control objective is reached because the solutions have an almost oscillatory behaviour and thereby strong persistence of the species is achieved.

2) *Asymptotic stability*: Consider any solution  $(a(t), b(t), \bar{s}(t), \bar{\xi}(t))$  of (35) with any initial condition such that  $a(0) + p_\delta(0) = \hat{s}(0) > 0$ ,  $a(0) + p_\delta(0) - \bar{s}(0) = s(0) > 0$ . Let us prove that this solution converges to the origin. We deduce from Lemma 1 and Lemma 4 that for all  $t \geq 0$  the solution is defined and satisfies  $a(t) + p_\delta(t) - \bar{s}(t) = s(t) > 0$ . Furthermore, the function  $W(a, b, \bar{s}, \bar{\xi})$  is positive definite and (34) implies that it is non-increasing along the trajectories. Since the function  $W(a, b, \bar{s}, \bar{\xi})$  does if one of its arguments goes to the infinity, we deduce that there exists a positive real number  $l > 0$  such that, for all  $t \geq 0$ ,

$$|a(t)| + |\bar{s}(t)| + \|\bar{\xi}(t)\| + \|b(t)\| \leq l. \quad (39)$$

We can apply now the LaSalle Invariance Principle extended to time-varying periodic systems (see for instance [13, Sec 5.4] or [6]). Since the solution of the system (35) we consider belongs to the compact set  $\mathcal{E} = \{(a, b, \bar{s}, \bar{\xi}) \in \mathbb{R}^{2N+2} : |a| + |\bar{s}| + \|b\| + \|\bar{\xi}\| \leq l\}$  for all  $t \geq 0$ , we deduce from (34) and the LaSalle Invariance Principle that this solution converges to the largest invariant set of (35) contained in the set

$$\mathcal{F} = \{(a, b, \bar{s}, \bar{\xi}) \in \mathbb{R}^{2N+2} : a = 0, \bar{s} = 0, \|\bar{\xi}\| \leq l, \|b\| \leq l\}.$$

To determine what is this set, we analyze the behavior of any trajectory  $(a_e, b_e, \bar{s}_e, \bar{\xi}_e)$  of (35) such that, for all  $t \geq 0$ , the inequality (39) is satisfied and  $a_e(t) = 0$ ,  $\bar{s}_e(t) = 0$ . From (35), we deduce that, for all  $t \geq 0$ ,

$$\sum_{i=1}^N b_{ie}(t) \mu'_i(p_\delta(t)) = 0, \quad \dot{b}_{je}(t) = 0, \quad j = 1, \dots, N. \quad (40)$$

It follows that, for all  $t \geq 0$ , the equality  $\sum_{j=1}^N b_{je}(0) \mu'_j(p_\delta(t)) = 0$  holds. From Assumption H3, we deduce that  $b_e(0) = 0$ . Next, using the fact that  $\bar{s}_e(t) = 0$  and  $\hat{\xi}_e(t) = \xi_{jr}(t)$  for all  $t \geq 0$ , we deduce from the two last equations of (35) that, for all  $t \geq 0$ ,

$$\sum_{j=1}^N \mu_j(p_\delta(t)) e^{\xi_{jr}(t)} \left(1 - e^{-\bar{\xi}_{je}(t)}\right) = 0, \quad \bar{\xi}_{ie}(t) = \bar{\xi}_{ie}(0), \quad \forall i = 1, \dots, N \quad (41)$$

and therefore, for all  $t \geq 0$ ,

$$\sum_{j=1}^N \mu_j(p_\delta(t)) \exp\left(\frac{\delta}{T} \int_{t-\frac{T}{\delta}}^t \int_r^t [\mu_j(p_\delta(\ell)) - D_*] d\ell dr\right) \left(1 - e^{-\bar{\xi}_{je}(0)}\right) = 0. \quad (42)$$

Next, using Assumption H4, one can prove that  $\bar{\xi}_e(t) = 0$ , for all  $t \geq 0$  when  $\delta$  is chosen sufficiently small (see [7], [8] for this proof).

Hence, we know that, for all  $t \geq 0$ ,  $a_e(t) = 0$ ,  $\bar{s}_e(t) = 0$ ,  $b_e(t) = 0$ ,  $\bar{\xi}_e(t) = 0$ . Therefore  $\mathcal{F} = \{(0, 0, 0, 0)\}$ . It follows that the largest invariant set of (35) contained in  $\mathcal{F}$  is  $\{(0, 0, 0, 0)\}$  and therefore the trajectory  $(a(t), b(t), \bar{s}(t), \bar{\xi}(t))$  we have considered converges to the origin.

We can conclude that our output feedback globally asymptotically stabilizes the positive reference trajectory of (1) defined in (7).

#### IV. EXAMPLE

We consider the system (1) with  $N = 2$ .  $\mu_1(s) = \frac{4s}{1+s}$ ,  $\mu_2(s) = \frac{64}{9} \left(\frac{s}{1+s}\right)^2$ . This system satisfies Assumptions H1 to H4 with  $p(t) = \frac{2+\sin(t)}{2-\sin(t)}$ ,  $T = 2\pi$ ,  $\mu_B = \frac{64}{9}$ ,  $m_1 = \frac{1}{3}$ ,  $m_2 = 3$ ,  $D_* = 2$ . This leads us to introduce the constants  $\delta = \frac{1}{36}$ ,  $\epsilon = \frac{3}{512}$ ,  $K = 1$  which satisfy (36). The reference state trajectory we consider is  $s_r(t) = p(\delta t)$ ,  $\xi_{1r}(t) = -\frac{1}{\delta} \cos(\delta t)$ ,  $\xi_{2r}(t) = -\frac{1}{\delta} [16 \cos(\delta t) + \sin(2\delta t)]$ . We simulate the system endowed with its observer when the control is the one provided by Theorem 1. With  $b = (\hat{\xi}_1 - \xi_{1r}(t), \hat{\xi}_2 - \xi_{2r}(t))$ ,  $R(b) = \sqrt{(\hat{\xi}_1 - \xi_{1r}(t))^2 + (\hat{\xi}_2 - \xi_{2r}(t))^2 + 1}$ , this system, in the  $\xi$ -coordinates, is

$$\left\{ \begin{array}{l} \dot{s} = -2s + 2p_\delta(t) + \frac{4\delta \cos(\delta t)}{[2-\sin(\delta t)]^2} + \frac{\epsilon b_1}{R(b)} \frac{4s}{1+s} e^{\hat{\xi}_1} + \frac{64\epsilon b_2}{9R(b)} \left(\frac{s}{1+s}\right)^2 e^{\hat{\xi}_2} \\ \quad + \frac{4s}{1+s} \left(e^{\hat{\xi}_1} - e^{\xi_1}\right) + \frac{64}{9} \left(\frac{s}{1+s}\right)^2 \left(e^{\hat{\xi}_2} - e^{\xi_2}\right) \\ \quad - \frac{\epsilon b_1}{R(b)(1+\hat{s})} \left[\frac{1}{1+p_\delta(t)} + \frac{1}{1+s}\right] - \frac{64\epsilon b_2}{9R(b)(1+\hat{s})^2} \left[\frac{p_\delta(t)+\hat{s}+2p_\delta(t)\hat{s}}{(1+p_\delta(t))^2} + \frac{s+\hat{s}+2s\hat{s}}{(1+s)^2}\right], \\ \dot{\hat{\xi}}_1 = \frac{4s}{1+s} - 2, \\ \dot{\hat{\xi}}_2 = \frac{64}{9} \left(\frac{s}{1+s}\right)^2 - 2, \\ \dot{\hat{s}} = -2\hat{s} + 2p_\delta(t) + \frac{4\delta \cos(\delta t)}{[2-\sin(\delta t)]^2} - \frac{4\epsilon b_1}{R(b)(1+p_\delta(t))(1+\hat{s})} - \frac{64\epsilon b_2}{9R(b)} \frac{p_\delta(t)+\hat{s}+2p_\delta(t)\hat{s}}{(1+p_\delta(t))^2(1+\hat{s})^2} \\ \dot{\hat{\xi}}_1 = (\hat{s} - s) \frac{4s}{1+s} e^{\hat{\xi}_1} + \frac{4s}{1+s} - 2, \\ \dot{\hat{\xi}}_2 = \frac{64}{9} (\hat{s} - s) \left(\frac{s}{1+s}\right)^2 e^{\hat{\xi}_2} + \frac{64}{9} \left(\frac{s}{1+s}\right)^2 - 2 \end{array} \right. , \quad (43)$$

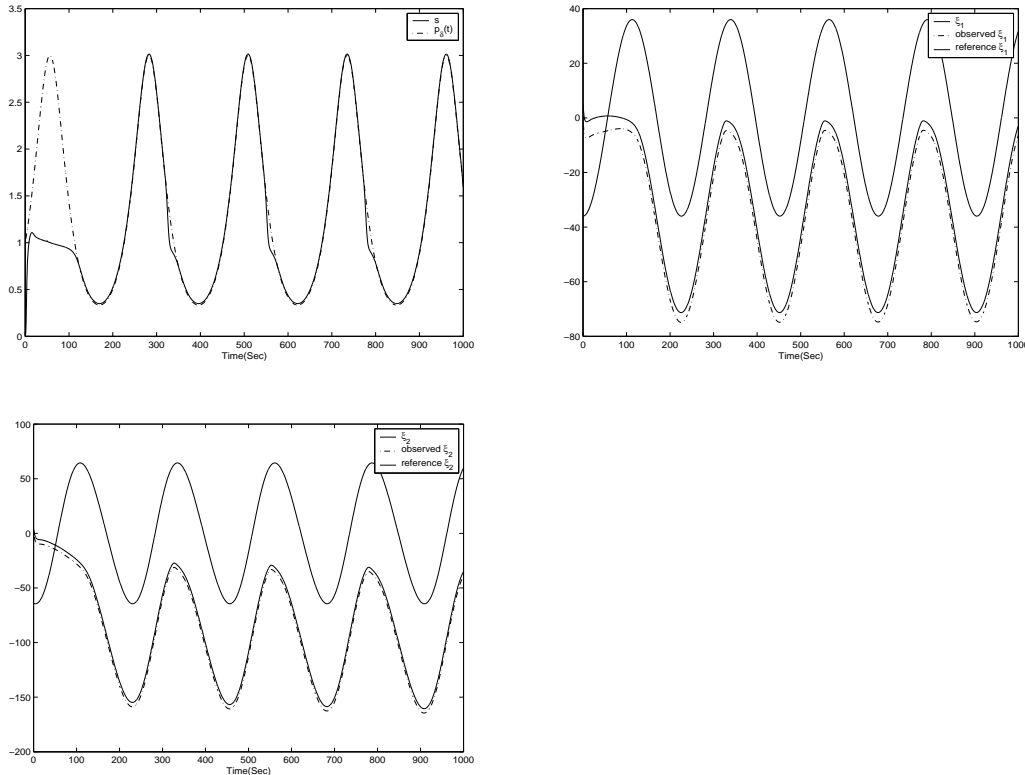
and the control law is

$$\begin{aligned} s_{in}(t, s, h) = & p_\delta(t) + \frac{\dot{p}_\delta(t)}{2} + \left(\frac{\epsilon b_1}{R(b)} + 1\right) \frac{2s}{1+s} e^{\hat{\xi}_1} + \frac{32}{9} \left(\frac{\epsilon b_2}{R(b)} + 1\right) \left(\frac{s}{1+s}\right)^2 e^{\hat{\xi}_2} \\ & - \frac{\epsilon b_1}{2R(b)(1+\hat{s})} \left[\frac{1}{1+p_\delta(t)} + \frac{1}{1+s}\right] - \frac{32\epsilon b_2}{9R(b)(1+\hat{s})^2} \left[\frac{p_\delta(t)+\hat{s}+2p_\delta(t)\hat{s}}{(1+p_\delta(t))^2} + \frac{s+\hat{s}+2s\hat{s}}{(1+s)^2}\right]. \end{aligned} \quad (44)$$

We give below figures which represent trajectories of the system (44):

#### V. CONCLUSIONS

We solved the problem of globally asymptotically stabilizing a periodic positive reference trajectory of a model of chemostat with an arbitrary number of species when the substrate



concentration is the only variable available by measurement. Much remains to be done. In particular, by constructing strict Lyapunov functions using the technique of [6, Chapter 8], we shall determine a robustness margin for the closed-loop system.

**Acknowledgment.** This work is supported in part by a seed grant from Poly and NYU and by NSF grant DMS-0906659. We thank Yu Jiang for helping us with the Matlab simulations in the revised version of this paper.

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## APPENDIX

From (18), (28) and (33), we deduce that the feedback  $s_{in}$  we have constructed has the analytic expression (37). Using Assumption H1, one can prove that

$$\left| \frac{1}{R(b)} \sum_{j=1}^N b_j \left[ \int_0^1 \mu'_j((1-\ell)p_\delta(t) + \ell\hat{s})d\ell + \frac{1}{K} \int_0^1 \mu'_j((1-\ell)\hat{s} + \ell s)d\ell \right] \right| \leq \left(1 + \frac{1}{K}\right) \sqrt{N} \mu_B . \quad (45)$$

Moreover, for all  $s \geq 0$ , the term  $\sum_{j=1}^N \mu_j(s)e^{\hat{\xi}_j} + \frac{\epsilon}{KR(b)} \sum_{j=1}^N b_j \mu_j(s)e^{\hat{\xi}_j}$  is non-negative when  $K \geq \epsilon$ . It follows that

$$s_{in}(t, s, h) \geq p_\delta(t) + \frac{1}{D_*} \dot{p}_\delta(t) - \frac{1}{D_*} \left(1 + \frac{1}{K}\right) \epsilon \sqrt{N} \mu_B . \quad (46)$$

From Assumption H2, we deduce that  $p_\delta(t) + \frac{1}{D_*} \dot{p}_\delta(t) \geq m_1 - \frac{\delta m_2}{D_*}$  and

$$s_{in}(t, s, h) \geq m_1 - \frac{\delta m_2}{D_*} - \left(1 + \frac{1}{K}\right) \frac{\epsilon \sqrt{N} \mu_B}{D_*} . \quad (47)$$

Finally (36) allows us to conclude.