Design of Continuous-Discrete Observers for Time-Varying Nonlinear Systems*

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Abstract

We present a new design for continuous-discrete observers for a large class of continuous time nonlinear time-varying systems with discrete time measurements. Using the notion of cooperative systems, we show that the solutions of the observers converge to the solutions of the original system, under conditions on the nonlinear terms and on the largest sampling interval. Our conditions are given by explicit expressions.

Key words: observer, continuous-discrete observers, cooperative system, Metzler matrix

1 Introduction

In real world applications, the state variables may be difficult to measure. Such applications can often be modeled using systems with outputs. Then one builds an observer for the state such that the observation error between the observer value and the state value converges to 0 as time goes to ∞. Much of the observers literature is under continuous measurements. See, e.g., Zemouche *et al.* (2008), which gives observers under continuous state measurements, by expressing the differential equation satisfied by the estimation error in terms of a linear parameter varying system.

However, in many engineering applications, measurements are collected at discrete times. This produces continuous-discrete systems, where the dynamics are continuous time but the output measurements are only available at discrete instants. There is a large literature, spanning over 40 years, on ways to build observers for continuous-discrete systems. See, e.g., Jazwinski (2007), which used a continuous-discrete Kalman filter to solve a filtering problem for stochastic continuous-discrete time systems.

The high gain observer in Gauthier *et al.* (1992) was adapted to continuous-discrete systems in Deza *et al.* (1992), where the correction gain of the impulsive correction is obtained by integrating a continuous-discrete time Riccati equation. The robustness of observers with respect to discretization was studied in Arcak and Nesic (2004). See also Ahmed-Ali *et al.* (2013*b*); Farza *et al.* (2013); Karafyllis and Kravaris (2009) for observers based on output predictors and Andrieu and Nadri (2010); Deza *et al.* (1992); Hammouri *et al.* (2006); Mazenc and Dinh (2013, 2014); Tellez-Anguiano *et al.* (2012); Karafyllis and Kravaris (2012); and see Ahmed-Ali *et al.* (2013*a*), which presents results that allow delayed and sampled measurements. The work Ahmed-Ali *et al.* (2009) builds continuous-discrete observers for nonlinear systems, where the input acts on the system to satisfy a persistent excitation condition, while Nadri and Hammouri (2003) covers systems with known inputs and which are linear in the state. The work Karafyllis and Kravaris (2009) shows that if a system admits a suitable continuous time observer and the observer satisfies certain robustness properties, then one can augment the observer by a new output predictor system to produce a continuous-discrete observer. Also, Karafyllis and Kravaris (2009) shows how this observer augmentation process applies to key classes of linear and triangular globally Lipschitz systems. See Remark 3 below

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for more discussions on Karafyllis and Kravaris (2009); and Astorga *et al.* (2002) for continuous-discrete observers for an important model of emulsion polymerization reactors.

Here, we revisit Andrieu and Nadri (2010). We present a new construction of continuous-discrete observers (which are also called hybrid observers in the literature) for continuous time Lipschitz systems with discrete time measurements. Following the approach in Andrieu and Nadri (2010); Deza *et al.* (1992), our continuous-discrete observer is obtained in two steps. First, when no measurement is available, the state estimate is computed by integrating the model. Then, when a measurement occurs, the observer makes an impulsive correction to the estimate. The work Andrieu and Nadri (2010); Dinh *et al.* (2015) used this type of algorithm to show that when no measurement occurs, the estimation error is a solution to an unknown linear parameter varying system. This gave a continuous-discrete analog of the continuous time measurement approach from Zemouche *et al.* (2008), and makes it possible to build a set that is guaranteed to contain all relevant solutions for all nonnegative times. Using this set, certain correction terms are designed to ensure that the estimation error asymptotically converges to zero. However, Andrieu and Nadri (2010); Dinh *et al.* (2015) find the set by integrating a system with commutation, which does not lead to an explicit analytic expression. This may be an obstruction to using this type of approach in applications. Here, we use tools that are inspired by Haddad *et al.* (2010); Cacace *et al.* (2012); Mazenc and Dinh (2013); Raissi *et al.* (2012). We obtain analytical methods for constructing sets that are guaranteed to contain the relevant trajectories. Our results are strong and may be better suited to applications, since we allow nonlinearities in the systems and because we prove robustness to perturbations in the sampling schedule.

In the next section, we provide definitions. In Section 3, we present our new results on framers, which are of independent interest. In Section 4, we use our new results on framers to prove our theorem on continuous-discrete observers. Our closed form formulas for the framers make it possible to check the assumptions using linear matrix inequalities. We illustrate our main result in Section 5, using a motor dynamics and a pendulum system, which show how our approach can lead to a much larger maximal allowable measurement stepsize than was reported in Dinh *et al.* (2015). In Section 6, we summarize the value added by our work and suggest possible topics for follow-up research.

For a tutorial paper on the theory of continuous-discrete observers that states results from this and other recent papers (without giving their proofs and also without the examples we provide below) and also contains a generalization of Lemma 3 below, see Mazenc *et al.* (2015, to appear).

2 Notation, Definitions, and Basic Result

Throughout the sequel, we omit arguments of functions when the arguments are clear from the context. We set $\mathbb{N} = \{1, 2, \ldots\}$. For any k and n in $\mathbb{N}$, the $k \times n$ matrix all of whose entries are 0 will also be denoted by 0, and we use $A = [a_{i,j}]$ to indicate that an arbitrary matrix $A \in \mathbb{R}^{k\times n}$ has $a_{i,j}$ in its ith row and jth column for each $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, n\}$. The usual Euclidean norm of vectors, and the induced norm of matrices, of any dimensions are denoted by $|\cdot|$, and I is the identity matrix in the dimension under consideration. All inequalities and maxima must be understood to hold *componentwise*, i.e., if $A = [a_{i,j}]$ and $B = [b_{i,j}]$ are matrices of the same dimensions, then we use $A \le B$ to mean that $a_{i,j} \le b_{i,j}$ for all i and j, and $\max\{A, B\}$ is the matrix $C = [c_{i,j}]$ where $c_{i,j} = \max\{a_{i,j}, b_{i,j}\}$ for all i and j. A square matrix is called *cooperative* or *Metzler* provided all of its off-diagonal entries are nonnegative. Recall that the *Schur complement* of a symmetric matrix of the form

$$X = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$$

with an invertible matrix A is $S = C - B^\top A^{-1} B$, where $\top$ means the transpose. The following is well known: X is positive definite if and only if A and S are both positive definite. For each $r \in \mathbb{N}$ and each function $\mathcal{F} : [0, \infty) \to \mathbb{R}^r$, we set $\mathcal{F}(t_-) = \lim_{s\to t, s<t} \mathcal{F}(s)$ for all $t > 0$. For any matrices A and B in $\mathbb{R}^{n\times n}$, we use $A \preccurlyeq B$ (resp., $A \succ B$) to mean that $X^\top(A - B)X \le 0$ for all $X \in \mathbb{R}^n$ (resp., $X^\top(A - B)X > 0$ for all $X \in \mathbb{R}^n \setminus \{0\}$). Therefore, $\succeq$ has a different meaning from the partial order $\ge$ on matrices.

A system $\dot{x}(t) = f(t, x(t))$ whose solution is uniquely defined on $[t_0, \infty)$ for each initial condition $x(t_0)$ and each $t_0 \ge 0$ is called *nonnegative* provided that for each initial condition satisfying $x(t_0) \ge 0$, the solution $x(t)$ is nonnegative for all $t \ge t_0$. The following lemma is a direct consequence of (Haddad *et al.*, 2010, Proposition 2.2):

Lemma 1 *Consider any system of the form*

$$\dot{z}(t) = \mathfrak{A}(t)z(t) + \mathfrak{B}(t) \tag{1}$$

with state space $\mathbb{R}^n$*, where* $\mathfrak{A} : \mathbb{R} \to \mathbb{R}^{n\times n}$ *and* $\mathfrak{B} : \mathbb{R} \to \mathbb{R}^n$ *are continuous. Assume that for all* $t \ge 0$*, the matrix* $\mathfrak{A}(t)$ *is Metzler and* $\mathfrak{B}(t) \ge 0$*. Then (1) is nonnegative.*

3 Preliminary Result on Framers

In this section, we present preliminary results on framers for linear systems that we use in the next section to design our observers for nonlinear systems. We consider any linear time-varying system of the form

$$\dot{x}(t) = \mathfrak{M}(t)x(t) \tag{2}$$

with state space $\mathbb{R}^n$, where all entries of $\mathfrak{M} : [0, \infty) \to \mathbb{R}^{n\times n}$ are continuous. Let $\varrho : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ denote the fundamental solution of (2). Then, $\frac{\partial}{\partial t}\varrho(t, t_0) = \mathfrak{M}(t)\varrho(t, t_0)$ and $\varrho(t_0, t_0) = I$ hold for all $t_0 \ge 0$ and $t \ge t_0$. In this section, we provide componentwise lower and upper bounds for $\Gamma(t) = \varrho(t, 0)$. Notice for later use that the unique solution $\phi(\cdot, x_0)$ of the initial value problem

$$(\partial\phi/\partial t)(t, x_0) = \mathfrak{M}(t)\phi(t, x_0),\ \phi(0, x_0) = x_0 \tag{3}$$

satisfies $\phi(t, x_0) = \Gamma(t)x_0$ for all $t \ge 0$ and $x_0 \in \mathbb{R}^n$.

3.1 Bounds for Cooperative Linear Systems

We first present a preliminary result on framers, which we use in the next subsection to prove our main result on framers. Throughout this subsection, we assume:

Assumption 1 *There are two constant Metzler matrices* $\overline{\mathfrak{M}} \in \mathbb{R}^{n\times n}$ *and* $\underline{\mathfrak{M}} \in \mathbb{R}^{n\times n}$ *such that*

$$\underline{\mathfrak{M}} \leq \mathfrak{M}(t) \leq \overline{\mathfrak{M}} \quad (4)$$

hold for all $t \geq 0$. *Also,* $\mathfrak{M}$ *is continuous.*

We can then prove:

Lemma 2 *If (2) satisfies Assumption 1, then* $\exp(\underline{\mathfrak{M}}t) \leq \Gamma(t) \leq \exp(\overline{\mathfrak{M}}t)$ *holds for all* $t \geq 0$.

Proof: We introduce the functions

$$\overline{\psi}(t,x) = e^{\overline{\mathfrak{M}}t}x \quad \text{and} \quad \underline{\psi}(t,x) = e^{\underline{\mathfrak{M}}t}x \quad (5)$$

which are defined on $[0,\infty) \times \mathbb{R}^n$. Since $\overline{\mathfrak{M}}$ and $\underline{\mathfrak{M}}$ are Metzler, Lemma 1 implies that $\dot{x} = \overline{\mathfrak{M}}x$ and $\dot{x} = \underline{\mathfrak{M}}x$ are nonnegative systems. Therefore, $\exp(\overline{\mathfrak{M}}t)e_i \geq 0$ and $\exp(\underline{\mathfrak{M}}t)e_i \geq 0$ hold for each standard basis element e_i for $1 \leq i \leq n$ and all $t \geq 0$. Hence,

$$\overline{\psi}(t,x_0) \geq 0\ ,\ \underline{\psi}(t,x_0) \geq 0\ ,\ \text{and}\ \phi(t,x_0) \geq 0 \quad (6)$$

hold for all $t \geq 0$ and $x_0 \geq 0$, where the nonnegativity of the flow map $\phi(t,x_0)$ for (2) is also from Lemma 1.

Fix any componentwise nonnegative $x_0 \in \mathbb{R}^n$. Then $\overline{\rho}(t,x_0) = \overline{\psi}(t,x_0) - \phi(t,x_0)$ satisfies

$$\begin{aligned} \dot{\overline{\rho}}(t,x_0) &= \overline{\mathfrak{M}}\,\overline{\psi}(t,x_0) - \mathfrak{M}(t)\phi(t,x_0) \\ &= \overline{\mathfrak{M}}\,\overline{\rho}(t,x_0) + \big(\overline{\mathfrak{M}} - \mathfrak{M}(t)\big)\phi(t,x_0) \end{aligned} \quad (7)$$

for all $t \geq 0$. Assumption 1 and the fact that $\phi(t,x_0) \geq 0$ give $(\overline{\mathfrak{M}} - \mathfrak{M}(t))\phi(t,x_0) \geq 0$ for all $t \geq 0$. Since $\overline{\mathfrak{M}}$ is Metzler and $\overline{\rho}(0,x_0) \geq 0$, we can apply Lemma 1 with $\mathfrak{A} = \overline{\mathfrak{M}}$ and $\mathfrak{B}(t) = (\overline{\mathfrak{M}} - \mathfrak{M}(t))\phi(t,x_0)$ to get $\overline{\rho}(t,x_0) \geq 0$ for all $t \geq 0$. Next note that $\underline{\rho}(t,x_0) = \phi(t,x_0) - \underline{\psi}(t,x_0)$ satisfies

$$\begin{aligned} \dot{\underline{\rho}}(t,x_0) &= \mathfrak{M}(t)\phi(t,x_0) - \underline{\mathfrak{M}}\underline{\psi}(t,x_0) \\ &= \mathfrak{M}(t)\underline{\rho}(t,x_0) + \big(\mathfrak{M}(t) - \underline{\mathfrak{M}}\big)\underline{\psi}(t,x_0) \end{aligned} \quad (8)$$

for all $t \geq 0$. Arguing as we did to show that $\overline{\rho}(t,x_0) \geq 0$ proves that $\underline{\rho}(t,x_0) \geq 0$ for all $t \geq 0$. Thus,

$$\underline{\psi}(t,x_0) \leq \phi(t,x_0) \leq \overline{\psi}(t,x_0) \quad (9)$$

hold for all $t \geq 0$ and $x_0 \geq 0$. These inequalities can be written as $\exp(\underline{\mathfrak{M}}t)x_0 \leq \Gamma(t)x_0 \leq \exp(\overline{\mathfrak{M}}t)x_0$. Since they hold for each standard basis vector $x_o = e_i$ for $i = 1,2,\ldots,n$ and all $t \geq 0$, the result follows. □

3.2 Bounds for General Time-Varying Linear Systems

In this part, we consider the system (2) under the following much weaker assumption than Assumption 1:

Assumption 2 *The matrix valued function* $\mathfrak{M} : [0,\infty) \to \mathbb{R}^{n\times n}$ *is bounded and continuous.*

Assumption 2 provides functions $\mathfrak{K} : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $\mathfrak{L} : \mathbb{R} \to \mathbb{R}^{n\times n}$, a constant matrix $\overline{\mathfrak{L}} \geq 0$, and constant Metzler matrices $\overline{\mathfrak{K}} \in \mathbb{R}^{n\times n}$ and $\underline{\mathfrak{K}} \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} &\mathfrak{M}(t) = \mathfrak{K}(t) - \mathfrak{L}(t),\ \ 0 \leq \mathfrak{L}(t) \leq \overline{\mathfrak{L}}, \\ &\text{and}\ \ \underline{\mathfrak{K}} \leq \mathfrak{K}(t) \leq \overline{\mathfrak{K}} \end{aligned} \quad (10)$$

hold for all $t \geq 0$. The preceding decomposition can be found by replacing the entries of $\mathfrak{M}(t) = [m_{i,j}(t)]$ by $m_{i,j}(t) + \overline{B}$ for a big enough constant $\overline{B} > 0$ to produce the Metzler matrices $\mathfrak{K}(t)$, and then letting $\mathfrak{L} = \overline{\mathfrak{L}}$ be the constant matrix having $\overline{B}$ as each entry, and $\underline{\mathfrak{K}} = 0$. However, other decompositions of the type (10) exist. We prove:

Lemma 3 *Let the system (2) satisfy Assumption 2, and let* $\mathfrak{L}$, $\mathfrak{K}$, $\overline{\mathfrak{L}} \in \mathbb{R}^{n\times n}$, $\overline{\mathfrak{K}} \in \mathbb{R}^{n\times n}$, *and* $\underline{\mathfrak{K}} \in \mathbb{R}^{n\times n}$ *satisfy the preceding requirements. Define the* C^1 *functions* $\underline{\Gamma} : [0,\infty) \to \mathbb{R}^{n\times n}$ *and* $\overline{\Gamma} : [0,\infty) \to \mathbb{R}^{n\times n}$ *by*

$$\begin{aligned} &\underline{\Gamma}(t) = e^{\underline{\mathfrak{K}}t} + \tfrac{1}{2}\left[e^{(\overline{\mathfrak{K}}-\overline{\mathfrak{L}})t} - e^{(\overline{\mathfrak{K}}+\overline{\mathfrak{L}})t}\right] \ \ \text{and} \\ &\overline{\Gamma}(t) = \tfrac{1}{2}\left[e^{(\overline{\mathfrak{K}}+\overline{\mathfrak{L}})t} + e^{(\overline{\mathfrak{K}}-\overline{\mathfrak{L}})t}\right]\ . \end{aligned} \quad (11)$$

Then the $\underline{\Gamma}(t) \leq \Gamma(t) \leq \overline{\Gamma}(t)$ *hold for all* $t \geq 0$.

Proof: We introduce the matrices

$$\begin{aligned} &\mathfrak{H}(t) = \begin{bmatrix} \mathfrak{K}(t) & \mathfrak{L}(t) \\ \mathfrak{L}(t) & \mathfrak{K}(t) \end{bmatrix}\ ,\ \ \overline{\mathfrak{H}} = \begin{bmatrix} \overline{\mathfrak{K}} & \overline{\mathfrak{L}} \\ \overline{\mathfrak{L}} & \overline{\mathfrak{K}} \end{bmatrix}\ ,\ \ \text{and} \\ &\underline{\mathfrak{H}} = \begin{bmatrix} \underline{\mathfrak{K}} & 0 \\ 0 & \underline{\mathfrak{K}} \end{bmatrix}\ . \end{aligned} \quad (12)$$

Then for the flow map $\phi(t,x_0)$ of (2), the function $z(t) = (\phi(t,x_0), -\phi(t,x_0))$ taking values in $\mathbb{R}^{2n}$ satisfies $\dot{z}(t) = \mathfrak{H}(t)z(t)$ for all $t \geq 0$ and $x_0 \in \mathbb{R}^n$. This motivates our study of the system

$$\dot{\Lambda}(t) = \mathfrak{H}(t)\Lambda(t) \quad (13)$$

and its fundamental matrix $\mathcal{G} : [0,\infty)^2 \to \mathbb{R}^{2n\times 2n}$. Set $\Omega(t) = \mathcal{G}(t,0)$. Then, $\Omega(0) = I$, and the maximal solution of (13) for any initial state $\Lambda(0) = a \in \mathbb{R}^{2n}$ is $\Lambda(t) = \Omega(t)a$. We introduce the block decomposition

$$\Omega(t) = \begin{bmatrix} \Omega_{1,1}(t) & \Omega_{1,2}(t) \\ \Omega_{2,1}(t) & \Omega_{2,2}(t) \end{bmatrix}\ , \quad (14)$$

where $\Omega_{i,j}(t) \in \mathbb{R}^{n\times n}$ for all t, i, and j. Since we have $(\phi(t,x_0), -\phi(t,x_0))^\top$ is a solution of (13) that satisfies $(\phi(0,x_0), -\phi(0,x_0))^\top = (x_0, -x_0)^\top$ for all $x_0 \in \mathbb{R}^n$, we can use the existence and the uniqueness of solutions of (13) and the fact that $\phi(t,x_0) = \Gamma(t)x_0$ for all $t \geq 0$ and $x_0 \in \mathbb{R}^n$ to get

$$\begin{aligned} &\begin{bmatrix} \Gamma(t)x_0 \\ -\Gamma(t)x_0 \end{bmatrix} = \\ &\Omega(t)\begin{bmatrix} x_0 \\ -x_0 \end{bmatrix} = \begin{bmatrix} (\Omega_{1,1}(t) - \Omega_{1,2}(t))x_0 \\ (\Omega_{2,1}(t) - \Omega_{2,2}(t))x_0 \end{bmatrix} \end{aligned} \quad (15)$$

for all $t \geq 0$. Since the preceding equalities are satisfied for all $x_0 \in \mathbb{R}^n$, we deduce that

$$\Gamma(t) \;=\; \Omega_{1,1}(t) - \Omega_{1,2}(t) \tag{16}$$

holds for all $t \geq 0$. Moreover, for all $t \geq 0$, we have $\underline{\mathfrak{H}} \leq \mathfrak{H}(t) \leq \overline{\mathfrak{H}}$, and both $\underline{\mathfrak{H}}$ and $\overline{\mathfrak{H}}$ are Metzler. Hence, Lemma 2 applies to (13). It ensures that the inequalities $\exp(\underline{\mathfrak{H}}t) \leq \Omega(t) \leq \exp(\overline{\mathfrak{H}}t)$ are satisfied for all $t \geq 0$. Since the matrices $[I\ 0]$ and $[I\ 0]^\top$ are nonnegative, we deduce that

$$[I\ 0]\, e^{\underline{\mathfrak{H}}t} \begin{bmatrix} I \\ 0 \end{bmatrix} \;\leq\; [I\ 0]\, \Omega(t) \begin{bmatrix} I \\ 0 \end{bmatrix} \;\leq\; [I\ 0]\, e^{\overline{\mathfrak{H}}t} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{17}$$

holds for all $t \geq 0$. This and similar arguments give

$$\begin{aligned} & e^{\underline{\mathfrak{K}}t} \;\leq\; \Omega_{1,1}(t) \;\leq\; [I\ 0]\, e^{\overline{\mathfrak{H}}t} \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{and} \\ & 0 \;\leq\; \Omega_{1,2}(t) \;\leq\; [I\ 0]\, e^{\overline{\mathfrak{H}}t} \begin{bmatrix} 0 \\ I \end{bmatrix} \end{aligned} \tag{18}$$

for all $t \geq 0$. By combining (16) and (18), we deduce that

$$e^{\underline{\mathfrak{K}}t} - [I\ 0]\, e^{\overline{\mathfrak{H}}t} \begin{bmatrix} 0 \\ I \end{bmatrix} \;\leq\; \Gamma(t) \;\leq\; [I\ 0]\, e^{\overline{\mathfrak{H}}t} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

for all $t \geq 0$. The conclusion now follows from our lemma in the appendix, applied with $H = \overline{\mathfrak{H}}$. □

Remark 1 *Lemma 3 does not require $\mathfrak{M}$ to be known. Instead, we only need to know that $\mathfrak{M}$ is bounded and continuous, and we need to know the bounding matrices $\overline{\mathfrak{L}}$, $\overline{\mathfrak{K}}$, and $\underline{\mathfrak{K}}$. Assume that the only information we know about $\mathfrak{M}$ is that it is continuous, and that there are constant Metzler matrices $\underline{\mathfrak{M}}$ and $\overline{\mathfrak{M}}$ such that $\underline{\mathfrak{M}} \leq \mathfrak{M}(t) \leq \overline{\mathfrak{M}}$ hold for all $t \geq 0$. Then, Lemma 3 gives the best possible estimate of $\Gamma(t)$. To see why, notice that in that case, we can choose $\mathfrak{M}(t) = \mathfrak{K}(t)$, $\overline{\mathfrak{K}} = \overline{\mathfrak{M}}$, $\underline{\mathfrak{K}} = \underline{\mathfrak{M}}$, and $\mathfrak{L}(t) = \overline{\mathfrak{L}} = 0$. Then Lemma 3 gives $\exp(\underline{\mathfrak{K}}t) \leq \Gamma(t) \leq \exp(\overline{\mathfrak{K}}t)$ for all t. No better bounds can be obtained, because the cases $\Gamma(t) = \exp(\overline{\mathfrak{K}}t)$ and $\Gamma(t) = \exp(\underline{\mathfrak{K}}t)$ can occur. However, if $\mathfrak{M}(t)$ is not lower bounded by a Metzler matrix, then Lemma 3 can be conservative. For example, take*

$$\begin{aligned} & \mathfrak{M}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \;\; \mathfrak{K}(t) = \overline{\mathfrak{K}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ & \mathfrak{L}(t) = \overline{\mathfrak{L}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \end{aligned} \tag{19}$$

and $\underline{\mathfrak{K}} = 0$. Then, $\Gamma(t)$ is bounded, but Lemma 3 gives the unbounded framing functions $\overline{\Gamma}(t)$ and $\underline{\Gamma}(t)$.

4 Continuous-Discrete Observer

4.1 Main Result

We next propose a new solution to the problem of constructing exponentially stable continuous-discrete observers for nonlinear Lipschitz systems with discrete measurements. Our solution relies on Lemma 3.

Let $\underline{\nu} > 0$ and $\overline{\nu} > \underline{\nu}$ be any two constants, and fix any sequences $\{t_i\}$ and $\{\nu_i\}$ in $[0, \infty)$ such that

$$\begin{aligned} & t_0 = 0\ ,\ t_{i+1} = t_i + \nu_i\ , \\ & \text{and } \nu_i \in [\underline{\nu}, \overline{\nu}] \text{ for all } i \in \mathbb{N}. \end{aligned} \tag{20}$$

The t_i's will serve as the measurement times for the output in our nonlinear system

$$\begin{cases} \dot{x}_*(t) \;=\; A_* x_*(t) + \varphi_*(t, x_*(t)) + B_* u(t) \\ y_*(t) \;=\; C_* x_*(t_i) \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \in \mathbb{N} \end{cases} \tag{21}$$

with discrete measurements, where x_* and y_* are valued in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively for any n and p and where the input u is valued in $\mathbb{R}^q$ for any choice of $q \in \mathbb{N}$. The matrices A_*, B_*, and C_* are constant. Assume:

Assumption 3 *There is a constant invertible matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix $A = PA_*P^{-1}$ is Metzler. Also, $\varphi_* : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and the matrix valued function $(\partial\varphi_*/\partial x)(t, x)$ exists on $[0, \infty) \times \mathbb{R}^n$ and is bounded and continuous. Finally, u is piecewise continuous and bounded on the interval $[0, T]$ for each $T > 0$.*

Notice that we do not require $(\partial\varphi_*/\partial t)(t, x)$ to exist at any points. See Remark 3 for motivation for our decomposition $A_*x_*(t) + \varphi_*(t, x_*(t))$ on the right side of (21). When all eigenvalues of A_* are real, we can find P by taking A to be the Jordan canonical form of A_*; see also Section 5.2 for an application where the eigenvalues of A_* are not necessarily all real. We set

$$\begin{aligned} & \varphi(t, x) = P\varphi_*(t, P^{-1}x), \;\; C = C_*P^{-1}, \\ & \text{and } \; w(t, a, b) = \int_0^1 \tfrac{\partial \varphi}{\partial x}(t, r(b-a) + a)\mathrm{d}r\ . \end{aligned} \tag{22}$$

Then the Fundamental Theorem of Calculus (applied to the function $f(r) = \varphi(t, r(b-a)+a)$ on the interval $[0, 1]$) gives

$$\begin{aligned} & \varphi(t, b) - \varphi(t, a) \\ & = [\textstyle\int_0^1 (\partial\varphi/\partial x)(t, r(b-a) + a)\mathrm{d}r](b - a) \\ & = w(t, a, b)(b - a) \end{aligned}$$

for all $t \geq 0$, $a \in \mathbb{R}^n$, and $b \in \mathbb{R}^n$. Also, it follows from Assumption 3 that there are nonnegative constants $\overline{v}_{i,j}$ such that each entry of $w = [w_{i,j}]$ satisfies $w_{i,j}(t, a, b) \in [-\overline{v}_{i,j}, \overline{v}_{i,j}]$ for all $t \geq 0$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $i \in \{1, ..., n\}$, and $j \in \{1, ..., n\}$. Let $\overline{D} = \mathrm{diag}\{\overline{v}_{1,1}, ..., \overline{v}_{n,n}\} \in \mathbb{R}^{n\times n}$ and $\overline{V} = [\overline{v}_{i,j}] \in \mathbb{R}^{n\times n}$, and choose the functions

$$\begin{aligned} & \underline{\beta}(\rho) = e^{(A-\overline{D})\rho} + \tfrac{1}{2}\left[e^{(A+\overline{D})\rho} - e^{(A+2\overline{V}-\overline{D})\rho}\right] \\ & \text{and } \; \overline{\beta}(\rho) = \tfrac{1}{2}\left[e^{(A+2\overline{V}-\overline{D})\rho} + e^{(A+\overline{D})\rho}\right]\ . \end{aligned} \tag{23}$$

In terms of our bounds $\underline{\nu}$ and $\overline{\nu}$ from (20) and the preceding matrices, our final assumption is:

Assumption 4 *There exist a constant matrix $K \in \mathbb{R}^{n\times p}$, a constant $\kappa \in (0,1)$, and a constant symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ such that for each constant matrix $\beta \in \mathbb{R}^{n\times n}$ satisfying*

$$\underline{\beta}(\rho) \le \beta \le \overline{\beta}(\rho) \ \text{ for some } \rho \in [\underline{\nu}, \overline{\nu}] \ , \tag{24}$$

the matrix inequality

$$\beta^\top (I-KC)^\top Q(I-KC)\beta \preceq \kappa Q$$

is satisfied.

See the next subsection for ways to verify Assumption 4. We prove the following, where we use the convention that $\hat{x}_*(0^-) = \hat{x}_*(0)$, and where $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ denote the smallest and largest eigenvalues of Q, respectively.

Theorem 1 *Let the system (21) satisfy Assumptions 3-4 and choose the continuous-discrete observer*

$$\begin{cases} \dot{\hat{x}}_*(t) \ = \ A_*\hat{x}_*(t) + \varphi_*(t, \hat{x}_*(t)) + B_*u(t), \\ \qquad\qquad \text{for } t \in [t_i, t_{i+1}), \ i \ge 0 \\ \hat{x}_*(t_i) \ = \ \hat{x}_*(t_i^-) + P^{-1}K[y_*(t_i) - CP\hat{x}_*(t_i^-)], \\ \qquad\qquad \text{for } i \ge 0 \ . \end{cases} \tag{25}$$

Then, the dynamics for $\tilde{x}_* = \hat{x}_* - x_*$ for (21) is uniformly globally exponentially stable to 0. In fact,

$$\begin{aligned} &|\tilde{x}_*(t)| \le \\ &\left(\tfrac{1}{\sqrt{\kappa}}\,\left|P^{-1}\right|\left|P\right|\exp\big(\overline{\nu}(|A| + |\bar{V}|)\big)\sqrt{\tfrac{\lambda_{\max}}{\lambda_{\min}}}\right) \\ &\times e^{\ln(\kappa)t/(2\overline{\nu})}|\tilde{x}_*(0)| \end{aligned} \tag{26}$$

holds for all $t \ge 0$ and all initial states $\tilde{x}_*(0)$.

Proof: First note that Assumption 3 implies that the finite escape time phenomenon does not occur. The change of coordinates $x = Px_*$ and our definitions give

$$\begin{cases} \dot{x}(t) \ = \ Ax(t) + \varphi(t, x(t)) + Bu(t) \\ y_*(t) \ = \ Cx(t_i) \ . \end{cases} \tag{27}$$

with $B = PB_*$. In a similar way, the change of coordinates $\hat{x} = P\hat{x}_*$ transforms (25) into

$$\begin{cases} \dot{\hat{x}}(t) \ = \ A\hat{x}(t) + \varphi(t, \hat{x}(t)) + Bu(t), \\ \qquad\qquad t \in [t_i, t_{i+1}), \ i \ge 0 \\ \hat{x}(t_i) \ = \ \hat{x}(t_i^-) + K[Cx(t_i) - C\hat{x}(t_i^-)], \quad i \ge 0 \ . \end{cases} \tag{28}$$

Using $\tilde{x} = \hat{x} - x$, our choice of w in (22), and the Fundamental Theorem of Calculus, and recalling that $x(t)$ is continuous, we get $x(t_i) = x(t_i^-)$ for all i and therefore also the following observation error dynamics:

$$\begin{cases} \dot{\tilde{x}}(t) \ = \ A\tilde{x}(t) + w(t, x(t), \hat{x}(t))\tilde{x}(t), \\ \qquad\qquad t \in [t_i, t_{i+1}), \ i \ge 0 \\ \tilde{x}(t_i) \ = \ (I - KC)\tilde{x}(t_i^-), \quad i \ge 0. \end{cases} \tag{29}$$

Choose any solutions $\hat{x}(t)$ and $x(t)$ of (28) and (27), respectively, and set $V(t) = w(t, x(t), \hat{x}(t))$.

Let $D_V(t)$ be the diagonal matrix whose diagonal entries are the corresponding diagonal entries of $V(t)$ for all $t \ge 0$. Then, all of the diagonal entries of $V_N(t) = V(t) - D_V(t)$ are zero. Let $V_p(t) = \max\{V_N(t), 0\}$, $V_q(t) = \max\{V_N(t), 0\} - V_N(t)$, and $A_V(t) = A + D_V(t) + V_p(t)$ for all $t \ge 0$. Then $V_N = V_p - V_q$. Also, our choices of $\overline{V}$ and $\overline{D}$ imply that $0 \le V_q(t) \le \overline{V} - \overline{D}$ and $A - \overline{D} \le A_V(t) \le A + \overline{V}$ hold for all $t \ge 0$, so A_V has constant Metzler upper and lower bounds. For each integer $i \ge 0$ and sample time t_i, we next study the system

$$\begin{aligned} \dot{X}(t) &= \big(A + V(t+t_i)\big)X(t) \\ &= \big(A_V(t+t_i) - V_q(t+t_i)\big)X(t) \ . \end{aligned} \tag{30}$$

In terms of the fundamental solution $\mathfrak{J}_i$ for (30), the function $X(t) = \Xi_i(t)X_0$ is the unique solution for (30) starting at any vector $X(0) = X_0 \in \mathbb{R}^n$, where $\Xi_i(t) = \mathfrak{J}_i(t, 0)$ for each i. Apply Lemma 3 with $\mathfrak{K}(t) = A_V(t+t_i)$, $\mathfrak{L}(t) = V_q(t+t_i)$, $\underline{\mathfrak{K}} = A - \overline{D}$, $\overline{\mathfrak{K}} = A + \overline{V}$, and $\overline{\mathfrak{L}} = \overline{V} - \overline{D}$ to get:

$$\underline{\beta}(t) \le \Xi_i(t) \le \overline{\beta}(t) \ \text{ for all } t \ge 0 \ , \tag{31}$$

where $\underline{\beta}$ and $\overline{\beta}$ are defined in (23). Since the observation error dynamics for $\tilde{x}(t) = \hat{x}(t) - x(t) = P\tilde{x}_*(t)$ are

$$\begin{cases} \dot{\tilde{x}}(t) \ = \ [A + V(t)]\tilde{x}(t) \quad t \in [t_i, t_{i+1}), \ i \ge 0 \\ \tilde{x}(t_i) \ = \ (I - KC)\tilde{x}(t_i^-), \quad i \ge 0 \ , \end{cases} \ , \tag{32}$$

we can apply the preceding argument to the solution $X(t) = \tilde{x}(t_i + t)$ of (30) on the interval $[0, \nu_i)$ for each i. It follows from (31) that the matrix $\alpha_i = \mathfrak{J}_i(\nu_i, 0) \in \mathbb{R}^{n\times n}$ satisfies $\tilde{x}(t_{i+1}^-) = \alpha_i\tilde{x}(t_i)$ and $\underline{\beta}(\nu_i) \le \alpha_i \le \overline{\beta}(\nu_i)$ for each i. We next give the stability analysis for the discrete time system $\tilde{x}(t_{i+1}) = (I - KC)\alpha_i\tilde{x}(t_i)$.

Set $W(x) = x^\top Qx$. By applying Assumption 4 with $\beta = \alpha_i$ and $\rho = \nu_i$, and letting $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ be the smallest and largest eigenvalues of Q, respectively, we get $W(\tilde{x}(t_{i+1})) \le \kappa W(\tilde{x}(t_i))$ and so also $|\tilde{x}(t_i)| \le \kappa^{i/2}\sqrt{\lambda_{\max}/\lambda_{\min}}|\tilde{x}(0)|$ for all $i \ge 0$. Since $\kappa \in (0,1)$, we conclude that $\tilde{x}(t_i)$ converges exponentially to zero as $i \to \infty$. Also, using the bound $\bar{b} = |A| + |\bar{V}|$ on $|A + V(t)|$, we can integrate (32) to get $|\tilde{x}(t)| \le \exp(\bar{b}\overline{\nu})|\tilde{x}(t_i)|$ for all $t \in [t_i, t_{i+1})$ and all $i \ge 0$, where $\overline{\nu}$ is from (20). Moreover, for each integer $i \ge 0$ and each $t \in [t_i, t_{i+1}]$, we have $t \le t_{i+1} \le \overline{\nu}(i+1)$, so $i \ge (t/\overline{\nu}) - 1$. Recalling that $\kappa \in (0,1]$ and using our exponential decay estimate on $\tilde{x}(t_i)$, we get

$$\begin{aligned} |\tilde{x}(t)| \ &\le \ e^{\bar{b}\overline{\nu}}\kappa^{((t/\overline{\nu})-1)/2}\sqrt{\lambda_{\max}/\lambda_{\min}}|\tilde{x}(0)| \\ &= \ (e^{\bar{b}\overline{\nu}}/\sqrt{\kappa})e^{\ln(\kappa)t/(2\overline{\nu})}\sqrt{\lambda_{\max}/\lambda_{\min}}|\tilde{x}(0)| \end{aligned} \tag{33}$$

for all $t \in [t_i, t_{i+1})$ and $i \ge 0$. Since $\tilde{x}(t) = P\tilde{x}_*(t)$ for all $t \ge 0$, and since the right side of (33) does not depend on i, the result follows. □

Remark 2 *For all $\rho \ge 0$ and componentwise nonnegative matrices $M \in \mathbb{R}^{n\times n}$, we have $\exp(M\rho) \ge I$. Using this property, one can prove that*

$$I \ge \exp(-\overline{D}\rho) + \tfrac{1}{2}\Big(\exp\big(\overline{D}\rho\big) - \exp\big((2\overline{V} - \overline{D})\rho\big)\Big) \tag{34}$$

for all $\rho \geq 0$ (which can be checked by left multiplying (34) through by $\exp(\bar{D}\rho)$). If we now left multiply (34) by $\exp(A\rho)$, we get $\exp(A\rho) \geq \underline{\beta}(\rho)$. Similar reasoning shows that for all $\rho \geq 0$ we have

$$\underline{\beta}(\rho) \leq e^{A\rho} \leq \overline{\beta}(\rho). \tag{35}$$

Hence, Assumption 4 (applied with $\beta = e^{A\rho}$) implies for some ρ in $[\underline{\nu}, \overline{\nu}]$, we have $\exp(A^\top \rho)(I - KC)^\top Q(I - KC)\exp(A\rho) \preceq \kappa Q$. This implies that $(\exp(A\rho) - KC\exp(A\rho))^\top Q(\exp(A\rho) - KC\exp(A\rho)) - \kappa Q \prec 0$, so $\exp(A^\top \rho) - \exp(A^\top \rho)C^\top K^\top$ is Schur stable. This means that the pair $(\exp(A^\top \rho), \exp(A^\top \rho)C^\top)$ and so also $(\exp(A^\top \rho), C^\top)$ is discrete time controllable for some $\rho \in [\underline{\nu}, \overline{\nu}]$. This implies that a necessary condition for Assumption 4 to hold is that for some ρ in $[\underline{\nu}, \overline{\nu}]$ the pair $(e^{A\rho}, C)$ (or equivalently, $(e^{A_* \rho}, C_*)$) is detectable in the usual sense of linear discrete time systems.

Remark 3 *Our framers (31) for the fundamental matrix of the observation error dynamics made it possible to get closed form exponential decay estimates (26) on the observer error, which were not available in the state-of-the art results in Karafyllis and Kravaris (2009) that used a very different output predictor method from ours and applied the results to linear and globally Lipschitz triangular systems. When u is the zero function, we can rewrite the x_* dynamics in (21) as $\dot{x}_*(t) = \Psi_*(t, x_*(t))$, where $\Psi_*(t, x_*) = A_* x_* + \varphi_*(t, x_*)$. Then $(\partial \Psi_*/\partial x)(t, x)$ will be bounded and continuous, if $(\partial \varphi_*/\partial x)(t, x)$ is bounded and continuous, so we can replace the conditions on φ_* in Assumption 3 by the requirements that φ_* is continuous and that $(\partial \varphi_*/\partial x)(t, x)$ is bounded and continuous, and then simply select $A_* = 0$ and $P = I$. However, different nonzero choices of A_* in the decomposition $\Psi_*(t, x_*) = A_* x_* + \varphi_*(t, x_*)$ and different choices of P produce different conditions in our Assumption 4, so it may be helpful to consider different nonzero choices of A_*. See Section 5 for more discussions on the effects of different possible decompositions on the right side of (21).*

One aspect of the approach in Karafyllis and Kravaris (2009) is that the observer is not impulsive, and that the estimate is a continuous function of the time. This is made possible by using an output predictor. If our approach and the approach in Karafyllis and Kravaris (2009) produce the same limit when the measurement stepsize approaches zero, it is in general very difficult to say which one is more general than the other one. It would be interesting to find a way to use the tools of cooperative systems in this context, to obtain new conditions in the output predictor approach of Karafyllis and Kravaris (2009).

4.2 Checking Assumption 4

4.2.1 Detectability Based Result

There is a strong relationship between Assumption 4 and the detectability in the continuous time sense of the pair (A, C). To highlight this, let (A, C) be detectable and let us prove that Assumption 4 is satisfied, provided that the bounds on $\overline{V}$ and $\overline{D}$ and $\underline{\nu} > 0$ and $(\overline{\nu}/\underline{\nu}) - 1$ are small enough. In fact, our analysis makes it possible to explicitly determine bounds on the allowable values of $\overline{V}$, $\overline{D}$ $\underline{\nu}$ and $(\overline{\nu}/\underline{\nu}) - 1$. Detectability of (A, C) ensures the existence of a matrix L such that the matrix $A + LC$ is Hurwitz. This property ensures that there exist a symmetric positive definite matrix Q and a constant $c > 0$ such that that $Q(A + LC) + (A + LC)^\top Q \preceq -cQ$. Therefore, there is a constant $\rho_1 > 0$ such that for all constants $\rho \in (0, \rho_1]$, we have

$$\left[I + \rho(A + LC)^\top\right] Q \left[I + \rho(A + LC)\right] \preceq \left(1 - \frac{3}{4}c\rho\right) Q. \tag{36}$$

Since there is a smooth function $\mathfrak{G}$ such that $(I + \rho LC)e^{\rho A} = I + \rho(A + LC) + \rho^2 \mathfrak{G}(\rho)$ for all $\rho \in \mathbb{R}$, there is a constant $\rho_2 \in \left(0, \min\left\{\rho_1, \frac{1}{c}\right\}\right]$ such that for all $\rho \in (0, \rho_2]$,

$$\left[(I + \rho LC)e^{\rho A}\right]^\top Q(I + \rho LC)e^{\rho A} \preceq \left(1 - \frac{c\rho}{2}\right) Q. \tag{37}$$

Consider any matrix β such that $\underline{\beta}(\rho) \leq \beta \leq \overline{\beta}(\rho)$. Defining $\chi(\beta, \rho) = \beta - \exp(A\rho)$, inequality (37) can be rewritten as

$$\begin{aligned} &[(I + \rho LC)(\beta - \chi(\beta, \rho))]^\top Q(I + \rho LC)(\beta - \chi(\beta, \rho)) \\ &\preceq \left(1 - \tfrac{c\rho}{2}\right) Q. \end{aligned} \tag{38}$$

It is equivalent to

$$\begin{aligned} &\beta^\top (I + \rho LC)^\top Q(I + \rho LC)\beta \\ &\preceq -\chi(\beta, \rho)^\top (I + \rho LC)^\top Q(I + \rho LC)\chi(\beta, \rho) \\ &+2\chi(\beta, \rho)^\top (I + \rho LC)^\top Q(I + \rho LC)\beta + \left(1 - \tfrac{c\rho}{2}\right) Q. \end{aligned} \tag{39}$$

Since Q is symmetric and positive definite, the inequality

$$\begin{aligned} &2\chi(\beta, \rho)^\top (I + \rho LC)^\top Q(I + \rho LC)\beta \\ &\preceq a\chi(\beta, \rho)^\top (I + \rho LC)^\top Q(I + \rho LC)\chi(\beta, \rho) \\ &+\tfrac{1}{a}\beta^\top (I + \rho LC)^\top Q(I + \rho LC)\beta \end{aligned} \tag{40}$$

holds for any constant $a > 0$. The estimate (40) follows by applying the relation $(\sqrt{a}R - (1/\sqrt{a})S)^\top(\sqrt{a}R - (1/\sqrt{a})S) \succeq 0$ with the choices $R = \sqrt{Q}(I + \rho LC)\chi(\beta, \rho)$ and $S = \sqrt{Q}(I + \rho LC)\beta$ where $\sqrt{Q}$ denotes any symmetric positive definite matrix such that $\sqrt{Q}\sqrt{Q} = Q$. Combining (39) and (40), we obtain

$$\begin{aligned} &\beta^\top (I + \rho LC)^\top Q(I + \rho LC)\beta \preceq \left(1 - \tfrac{c\rho}{2}\right) Q \\ &+(a - 1)\chi(\beta, \rho)^\top (I + \rho LC)^\top Q(I + \rho LC)\chi(\beta, \rho) \\ &+\tfrac{1}{a}\beta^\top (I + \rho LC)^\top Q(I + \rho LC)\beta. \end{aligned} \tag{41}$$

Since $\rho_2 \in (0, \frac{1}{c}]$ and $\rho \in (0, \rho_2)$, we have $(4/(c\rho)) - 1 > 1$. Then, choosing $a = (4/(c\rho)) - 1$, we deduce from (41) that

$$\begin{aligned} &\beta^\top (I + \rho LC)^\top Q(I + \rho LC)\beta \preceq \left(1 - \tfrac{c\rho}{4}\right) Q \\ &+ \left(\tfrac{4}{c\rho} - 1\right) \chi(\beta, \rho)^\top (I + \rho LC)^\top \\ &\times Q(I + \rho LC)\chi(\beta, \rho). \end{aligned} \tag{42}$$

We will presently prove that the inequalities

$$\begin{aligned} &\left|\overline{\beta}(\rho) - e^{A\rho}\right| \leq \\ &\tfrac{\rho}{2}|2\overline{V} - \overline{D}|e^{(|A| + |2\overline{V} - \overline{D}|)\rho} + \tfrac{\rho}{2}|\overline{D}|e^{(|A| + |\overline{D}|)\rho} \end{aligned} \tag{43}$$

and

$$\left|\underline{\beta}(\rho) - e^{A\rho}\right| \leq \rho|\overline{D}|e^{(|A|+|\overline{D}|)\rho} + \rho|\overline{V} - \overline{D}|e^{(|A+2\overline{V}-\overline{D}|+2|\overline{V}-\overline{D}|)\rho} \tag{44}$$

hold for all $\rho > 0$; see below. Using (43)-(44), we get

$$|\chi(\beta,\rho)| \leq \rho k(\rho_2), \text{ where} \tag{45}$$

$$k(\rho) = n\max\left\{\left|\overline{V} - \frac{\overline{D}}{2}\right| e^{(|A|+|2\overline{V}-\overline{D}|)\rho} + \frac{|\overline{D}|}{2}e^{(|A|+|\overline{D}|)\rho}, |\overline{D}|e^{(|A|+|\overline{D}|)\rho} + |\overline{V}-\overline{D}|e^{(|A+2\overline{V}-\overline{D}|+2|\overline{V}-\overline{D}|)\rho}\right\}. \tag{46}$$

We deduce that there exist $\rho_3 \in (0,\rho_2]$ and a constant $c_* > 0$ such that for all $\rho \in (0,\rho_3]$, we have

$$\beta^\top(I + \rho LC)^\top Q(I + \rho LC)\beta \preceq \left(1 - \tfrac{c\rho}{4}\right) Q + c_*\rho(|\overline{V}| + |\overline{D}|)Q. \tag{47}$$

Therefore, when $|\overline{V}| + |\overline{D}| \leq \frac{c}{8c_*}$, the matrix inequality

$$\beta^\top(I + \rho LC)^\top Q(I + \rho LC)\beta \preceq \left(1 - \frac{c\rho}{8}\right) Q \tag{48}$$

is satisfied. Next, let $\rho \in [\underline{\nu}, \overline{\nu}]$ with $\overline{\nu} = \rho_3$ and $\underline{\nu} \in (0, \overline{\nu}]$ to be selected later. Then

$$\beta^\top(I + \rho LC)^\top Q(I + \rho LC)\beta \preceq \left(\kappa - \frac{c\underline{\nu}}{16}\right) Q, \tag{49}$$

with $\kappa = 1 - \frac{c\underline{\nu}}{16} \in (0,1)$. Let $K = -\underline{\nu}L$. Then (49) is equivalent to

$$\beta^\top[I-KC+(\rho-\underline{\nu})LC]^\top Q[I-KC+(\rho-\underline{\nu})LC]\beta \preceq \left(\kappa - \tfrac{c\underline{\nu}}{16}\right) Q. \tag{50}$$

Therefore, if the constant $(\overline{\nu}/\underline{\nu}) - 1$ is sufficiently small,

$$\beta^\top[I - KC]^\top Q[I - KC]\beta \preceq \kappa Q. \tag{51}$$

Thus Assumption 4 is satisfied. Now, let us establish (43) and (44). It follows from the Mean Value Theorem that for any matrices $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$, the inequality

$$\left|e^{X+Y} - e^X\right| \leq |Y|e^{|X|+|Y|} \tag{52}$$

is satisfied. Next, observing that

$$\overline{\beta}(\rho) - e^{A\rho} = \tfrac{1}{2}\left[e^{(A+2\overline{V}-\overline{D})\rho} - e^{A\rho}\right] + \tfrac{1}{2}\left[e^{(A+\overline{D})\rho} - e^{A\rho}\right]$$

and

$$\underline{\beta}(\rho) - e^{A\rho} = e^{(A-\overline{D})\rho} - e^{A\rho} + \tfrac{1}{2}\left[e^{(A+\overline{D})\rho} - e^{(A+2\overline{V}-\overline{D})\rho}\right]$$

we deduce from (52) that (43) and (44) are satisfied.

4.2.2 Linear Matrix Inequality Formalism

It can be convenient to check Assumption 4 using linear matrix inequalities. To see why, let $\underline{\nu} > 0$ and $\overline{\nu} \geq \underline{\nu}$ be any constants. We define $\overline{\beta}$ and $\underline{\beta}$ by (23), and we introduce the finite set $\mathcal{S}$ of matrices in $\mathbb{R}^{n\times n}$ that is defined by

$$\mathcal{S} = \left\{\omega \in \mathbb{R}^{n\times n} : \omega = [\omega_{i,j}] \text{ and } \omega_{i,j} \in \left\{\underline{\omega}_{i,j}, \overline{\omega}_{i,j}\right\} \text{ for all } i \text{ and } j\right\}, \text{ where} \tag{53}$$

$$\overline{\omega}_{i,j} = \sup_{l\in[\underline{\nu},\overline{\nu}]} \overline{\beta}_{i,j}(l) \text{ and } \underline{\omega}_{i,j} = \inf_{l\in[\underline{\nu},\overline{\nu}]} \underline{\beta}_{i,j}(l). \tag{54}$$

This allows us to rewrite Assumption 4 as a linear matrix inequality, as follows:

Proposition 1 *Assume that there are a positive definite symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and a matrix $W \in \mathbb{R}^{n\times p}$ such that for all $\omega \in \mathcal{S}$, the linear matrix inequality*

$$\begin{bmatrix} Q & (Q - WC)\omega \\ \omega^\top(Q - WC)^\top & Q \end{bmatrix} \succ 0 \tag{55}$$

holds. Then Assumption 4 holds with $K = Q^{-1}W$ and some constant κ.

Proof: Since the mappings $\underline{\beta}(\rho)$ and $\overline{\beta}(\rho)$ are continuous, and since eigenvalues are continuous functions of the entries of the corresponding matrix, there exists a constant $\epsilon \in (0,1)$ such that for all $\omega \in \mathcal{S}$, we have

$$\begin{bmatrix} (1-\epsilon)Q & (Q - WC)\omega \\ \omega^\top(Q - WC)^\top & (1-\epsilon)Q \end{bmatrix} \succ 0\,. \tag{56}$$

Let $\rho \in [\underline{\nu}, \overline{\nu}]$ and let β be such that the matrix inequality (24) holds for this value ρ. Note that β is in the closed convex hull of $\mathcal{S}$. Set $K = Q^{-1}W$. Since the matrix inequality (56) is linear in ω, it follows that

$$\begin{bmatrix} (1-\epsilon)Q & (Q - WC)\beta \\ \beta^\top(Q - WC)^\top & (1-\epsilon)Q \end{bmatrix} \succ 0\,. \tag{57}$$

Hence, the Schur complement of the matrix in (57) is

$$(1-\epsilon)Q - \tfrac{1}{1-\epsilon}\beta^\top(Q - WC)^\top Q^{-1}(Q - WC)\beta = (1-\epsilon)Q - \tfrac{1}{1-\epsilon}\beta^\top(I - KC)^\top Q(I - KC)\beta \succ 0,$$

so $\kappa Q \succ \beta^\top(I - KC)^\top Q(I - KC)\beta$ holds with the choice $\kappa = (1-\epsilon)^2$. Therefore, Assumption 4 holds. □

It is well known that if a linear system with continuous measurement has an observer, and if the measurement process is discretized with sufficiently small sampling time, then there is a continuous-discrete observer. This can be seen in our nonlinear case by noting that if a condition ensuring the existence of an observer in the case of continuous time measurement holds, then Assumption 4 holds when the ν_k's are small enough. This is made precise in the following proposition:

Proposition 2 *Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, and $\overline{V} \in \mathbb{R}^{n\times n}$ be matrices such that there exist a matrix $K_0 \in \mathbb{R}^{n\times p}$, a constant $\kappa_0 \in (0,1)$, and a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ such that*

$$(\alpha_0 - K_0C)^\top Q + Q(\alpha_0 - K_0C) \preceq -\kappa_0 Q \tag{58}$$

is satisfied for all matrices $\alpha_0 \in \mathbb{R}^{n\times n}$ such that $A - \overline{V} \leq \alpha_0 \leq A + \overline{V}$. Then there exists a constant $\nu_* > 0$ such that for all $\nu \in (0, \nu_*]$, Assumption 4 is satisfied with the choices $\underline{\nu} = \overline{\nu} = \nu$ and $K = \nu K_0$ and $\overline{D}$ is the diagonal matrix whose diagonal entries are identical to the corresponding diagonal entries of $\overline{V}$.

Proof: Let $\overline{\beta}$ and $\underline{\beta}$ be the $n \times n$ matrix valued functions defined in (23). We prove that there is a constant $\nu_* > 0$ such that the following holds: If $\nu \in (0, \nu_*]$ and $\beta \in \mathbb{R}^{n \times n}$ satisfy

$$\underline{\beta}(\nu) \leq \beta \leq \overline{\beta}(\nu), \tag{59}$$

then the linear matrix inequality

$$\beta^\top (I - KC)^\top Q(I - KC)\beta \preceq \kappa Q \tag{60}$$

holds with $\kappa = 1 - \frac{1}{2}\nu\kappa_0$, $\kappa \in (0,1)$, and $K = \nu K_0$.

To this end, first note that by using the change of variables $\omega = \alpha_0 - A$, we see that our assumptions in the proposition can be written as follows: For all ω that satisfy $-\overline{V} \leq \omega \leq \overline{V}$, the linear matrix inequality

$$[A - K_0 C + \omega]^\top Q + Q[A - K_0 C + \omega] \preceq -\kappa_0 Q \tag{61}$$

holds. Next, observe that we can determine two continuous functions φ_1 and φ_2 such that for all $\nu \in \mathbb{R}$,

$$\begin{aligned} \overline{\beta}(\nu) &= I + (A + \overline{V})\nu + \nu^2 \varphi_1(\nu) \text{ and} \\ \underline{\beta}(\nu) &= I + (A - \overline{V})\nu + \nu^2 \varphi_2(\nu). \end{aligned} \tag{62}$$

Let $\nu > 0$ be given and β satisfy (59), and set $\theta = (\beta - (I + \nu A))/\nu$. Using the equalities (62), we deduce that (59) is equivalent to $-\overline{V} + \nu\varphi_2(\nu) \leq \theta \leq \overline{V} + \nu\varphi_1(\nu)$. Also, (60) is equivalent to

$$\begin{aligned} &[I + \nu(A + \theta)]^\top (I - \nu K_0 C)^\top Q(I - \nu K_0 C) \\ &\times [I + \nu(A + \theta)] \preceq \left(1 - \tfrac{1}{2}\nu\kappa_0\right) Q. \end{aligned} \tag{63}$$

By viewing the left side of (63) as a polynomial in ν with matrix coefficients, collecting the coefficients of ν, and moving the terms involving ν^2, ν^3, and ν^4 to the right side, it follows that (63) can be written as

$$\begin{aligned} &Q - \nu C^\top K_0^\top Q - \nu Q K_0 C + \nu[A^\top + \theta^\top]Q \\ &+ \nu Q[A + \theta] \preceq \left(1 - \tfrac{1}{2}\nu\kappa_0\right) Q + \nu^2 \varphi_3(\nu), \end{aligned} \tag{64}$$

for a suitable continuous function φ_3. By subtracting Q from both sides and dividing through by ν, we deduce that (64) is equivalent to

$$\begin{aligned} &[A - K_0 C + \theta]^\top Q + Q[A - K_0 C + \theta] \\ &\preceq -\tfrac{1}{2}\kappa_0 Q + \nu\varphi_3(\nu). \end{aligned} \tag{65}$$

Using the continuity of the φ_i's and (61), we can find values $\nu_a > 0$ and $\nu_b > 0$ such that

$$[A - K_0 C + \theta]^\top Q + Q[A - K_0 C + \theta] \preceq -0.75\kappa_0 Q$$

holds for all θ that satisfy $-\overline{V} + \nu\varphi_2(\nu) \leq \theta \leq \overline{V} + \nu\varphi_1(\nu)$ and all $\nu \in (0, \nu_a)$, and such that

$$-0.75\kappa_0 Q \preceq -0.5\kappa_0 Q + \nu\varphi_3(\nu)$$

for all $\nu \in (0, \nu_b)$. Hence, (65) holds if (a) $0 < \nu \leq \min\{\nu_a, \nu_b)$ and (b) the matrix θ satisfies

$$-\overline{V} + \nu\varphi_2(\nu) \leq \theta \leq \overline{V} + \nu\varphi_1(\nu).$$

Therefore, we can satisfy our requirements by taking $\nu_* = \min\{\nu_a, \nu_b, 2\}$. □

5 Illustrations

5.1 Pendulum

As in Dinh *et al.* (2015), we first study the pendulum model

$$\dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = \sin(x_1(t)),\ y(t_k) = x_1(t_k) \tag{66}$$

with $\nu = \overline{\nu} = \underline{\nu}$, and x_1 and x_2 both real valued. Since x_1 is the position variable, it is realistic to assume that x_1 is measured. We apply Theorem 1. We verify Assumptions 3 and 4 with the choices $P = I$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\quad C = [1 \quad 0],\quad \varphi(t,x) = \begin{bmatrix} 0 \\ \sin(x_1) \end{bmatrix}, \tag{67}$$

$$\overline{V} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ \text{ and }\ \overline{D} = 0\,. \tag{68}$$

Assumption 3 holds with $P = I$ and $u = 0$, because A is Metzler and $\sin(\cdot)$ is Lipschitz. To check Assumption 4, first note that we have

$$\begin{aligned} A + 2\overline{V} = A + 2\overline{V} - \overline{D} &= \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \\ \text{and }\ e^{(A - \overline{D})\overline{\nu}} = e^{(A + \overline{D})\overline{\nu}} &= \begin{bmatrix} 1 & \overline{\nu} \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{69}$$

To facilitate finding the exponentials, we write

$$\begin{aligned} R(A + 2\overline{V})R^{-1} &= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix},\ \text{ where} \\ R^{-1} &= \begin{bmatrix} 1 & -1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \text{ and } R = \tfrac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{2} & 1 \\ -\sqrt{2} & 1 \end{bmatrix}. \end{aligned} \tag{70}$$

This gives

$$\begin{aligned} e^{(A + 2\overline{V})\overline{\nu}} \;=\; R^{-1} \begin{bmatrix} e^{\sqrt{2}\overline{\nu}} & 0 \\ 0 & e^{-\sqrt{2}\overline{\nu}} \end{bmatrix} R \;= \\ \begin{bmatrix} \frac{e^{\sqrt{2}\overline{\nu}} + e^{-\sqrt{2}\overline{\nu}}}{2} & \frac{e^{\sqrt{2}\overline{\nu}} - e^{-\sqrt{2}\overline{\nu}}}{2\sqrt{2}} \\ \frac{e^{\sqrt{2}\overline{\nu}} - e^{-\sqrt{2}\overline{\nu}}}{\sqrt{2}} & \frac{e^{\sqrt{2}\overline{\nu}} + e^{-\sqrt{2}\overline{\nu}}}{2} \end{bmatrix}. \end{aligned} \tag{71}$$

Therefore, the bounding functions from (23) are

$$\underline{\beta}(\overline{\nu}) = \frac{1}{2} \begin{bmatrix} 3 - \frac{e^{\sqrt{2}\overline{\nu}} + e^{-\sqrt{2}\overline{\nu}}}{2} & 3\overline{\nu} + \frac{e^{-\sqrt{2}\overline{\nu}} - e^{\sqrt{2}\overline{\nu}}}{2\sqrt{2}} \\ \frac{e^{-\sqrt{2}\overline{\nu}} - e^{\sqrt{2}\overline{\nu}}}{\sqrt{2}} & 3 - \frac{e^{\sqrt{2}\overline{\nu}} + e^{-\sqrt{2}\overline{\nu}}}{2} \end{bmatrix} \text{ and}$$

$$\overline{\beta}(\overline{\nu}) = \frac{1}{2} \begin{bmatrix} 1 + \frac{e^{\sqrt{2}\overline{\nu}} + e^{-\sqrt{2}\overline{\nu}}}{2} & \overline{\nu} + \frac{e^{\sqrt{2}\overline{\nu}} - e^{-\sqrt{2}\overline{\nu}}}{2\sqrt{2}} \\ \frac{e^{\sqrt{2}\overline{\nu}} - e^{-\sqrt{2}\overline{\nu}}}{\sqrt{2}} & 1 + \frac{e^{\sqrt{2}\overline{\nu}} + e^{-\sqrt{2}\overline{\nu}}}{2} \end{bmatrix}.$$

We now verify Assumption 4, using the linear matrix inequality condition from Proposition 1. Using the YALMIP package from Löfberg (2004) in MATLAB, combined with the solver SeDuMi from Sturm (1999), we can verify that (55) is satisfied for all $\omega \in \mathcal{S}$. In fact, when $\nu = 1.1$, condition (55) holds for all $\omega \in \mathcal{S}$ when we pick

$$K = [1.000,\ 0.8976]^\top \quad \text{and} \quad Q = \begin{bmatrix} 1534 & -5.7 \\ -5.7 & 10.5 \end{bmatrix} .$$

This is an important improvement, if we compare with Dinh *et al.* (2015), where the maximal measurement stepsize allowed is 0.668. Figure 1 shows our simulation of the observer using an integration algorithm of the model with a semi-implicit integration step of 0.001 with the initial conditions $x_1(0) = x_2(0) = 2$ and $\hat{x}_1(0) = \hat{x}_2(0) = 0$.

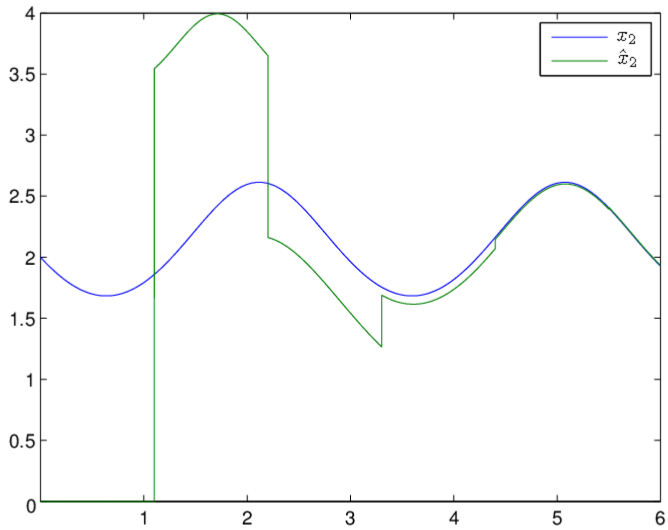

Fig. 1. Evolution with Time of the State Component x_2 and its Estimation $\hat{x}_2$.

5.2 *Robotic DC Motor*

The pendulum model in Section 5.1 had the form $\dot{x} = Ax + \varphi(t, x)$ where A is Metzler, so we were able to use the identity transformation $P = I$ in Assumption 3. To see how our Theorem 1 also applies in higher dimensional cases where a more complicated transformation is needed, consider the case of a single-link direct-drive manipulator actuated by a permanent magnet DC brush motor, which produces the following model from Dawson *et al.* (1994):

$$\begin{array}{l} M\ddot{q} + B\dot{q} + N\sin(q) = \mathcal{I},\ \ L\dot{\mathcal{I}} = V_e - R\mathcal{I} - K_B\dot{q}, \\ \text{where } M = \frac{J}{K_\tau} + \frac{mL_0^2}{3K_\tau} + \frac{M_0L_0^2}{K_\tau} + \frac{2M_0R_0^2}{5K_\tau}, \\ N = \frac{mL_0G}{2K_\tau} + \frac{M_0L_0G}{K_\tau}, \text{ and } B = \frac{B_0}{K_\tau} \end{array} \tag{72}$$

and where J is the rotor inertia, m is the mass of the link, M_0 is the mass of the load, L_0 is the length of the link, R_0 is the radius of the load, G is the gravitational constant, B_0 is the viscous friction coefficient at the joint, $q(t)$ is the position of the load (which is the angular motor position), $\mathcal{I}(t)$ is the motor armature current, the coefficient K_τ characterizes the electromagnetic conversion of armature current to torque, L is the armature inductance, R is the armature resistance, K_B is the back-emf coefficient, and V_e is the input current voltage. All of the constants in (72) are positive. See the schematic diagram in Figure 2 for (72).

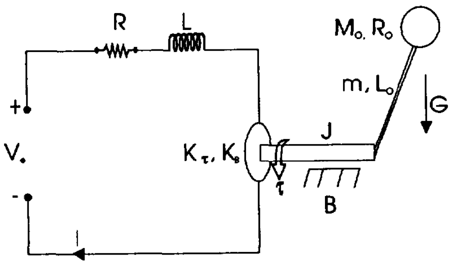

Fig. 2. Schematic from Dawson *et al.* (1994) for the Electromechanical System (72)

We rewrite (72) in the form

$$\begin{cases} \dot{x}_{1*} &= x_{2*} \\ \dot{x}_{2*} &= b_1x_{3*} - a_1\sin(x_{1*}) - a_2x_{2*} \\ \dot{x}_{3*} &= b_0U(t) - a_3x_{2*} - a_4x_{3*} \\ \ \ y_* &= x_{1*}\ , \end{cases} \tag{73}$$

where $x_{1*} = q$, $x_{2*} = \dot{q}$, $x_{3*} = \mathcal{I}$, $U = V_e$ is the control, $a_1 = N/M$, $a_2 = B/M$, $a_3 = K_B/L$, $a_4 = R/L$, $b_0 = 1/L$, and $b_1 = 1/M$, so all of the a_i's and b_i's in (73) are positive constants. We can write (73) in the form $\dot{x}_* = A_*x_* + \varphi_*(t, x_*)$ and $y_* = C_*x_*$, where

$$\begin{array}{l} A_* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -a_2 & b_1 \\ 0 & -a_3 & -a_4 \end{bmatrix}, \\ \varphi_*(t, x_*) = \begin{bmatrix} 0 \\ -a_1\sin(x_{1*}) \\ b_0U(t) \end{bmatrix}, \text{ and } C_* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^\top \end{array} \tag{74}$$

(but see Remark 4 for a discussion of other choices of A_* and φ_*). Then A_* is not Metzler. We now use ideas from (Mazenc and Bernard, 2011, Theorem 2) to transform (73) into a system that is covered by Theorem 1; (Mazenc and Bernard, 2011, Theorem 2) assumed that the matrix being transformed is Hurwitz, but the Hurwitzness is not needed for what follows. Set $\mathcal{D} = (a_2 - a_4)^2 - 4a_3b_1$. If $\mathcal{D} \geq 0$, then A_* has the eigenvalues

$$\begin{array}{l} \lambda_1 = 0,\ \lambda_2 = \frac{1}{2}\big(- (a_2 + a_4) + \sqrt{\mathcal{D}}\big), \text{ and} \\ \lambda_3 = \frac{1}{2}\big(- (a_2 + a_4) - \sqrt{\mathcal{D}}\big). \end{array} \tag{75}$$

On the other hand, when $\mathcal{D} < 0$, then A_* has the eigenvalues

$$\begin{array}{l} \lambda_1 = 0,\ \lambda_2 = \frac{1}{2}\big(- (a_2 + a_4) + i\sqrt{-\mathcal{D}}\big), \text{ and} \\ \lambda_3 = \frac{1}{2}\big(- (a_2 + a_4) - i\sqrt{-\mathcal{D}}\big)\ . \end{array} \tag{76}$$

The proof of (Mazenc and Bernard, 2011, Theorem 2) provides a time-varying transformation $x = P(t)x_*$ that transforms the system $\dot{x}_* = A_*x_* + \phi_*(t, x_*)$ into a new system $\dot{x} = Ax + \phi(t, x)$ where A is a constant Metzler matrix. In general, time-varying state transformations may transform the output and the measurement stepsize. Consider the case where the eigenvalues of A_* are distinct real numbers. In that case, we can take $A = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\} = \mathcal{M}^{-1} A_* \mathcal{M}$ and $P = \mathcal{M}^{-1}$, where $\mathcal{M}$ is a constant matrix whose columns are corresponding eigenvectors for the eigenvalues. If A_* has repeated real eigenvalues, then A is the usual real Jordan canonical form and we can again take P to be constant. Then A is Metzler, and the output is time invariant under this transformation. In the new variable x, the system satisfies Assumption 3 with $u = 0$. (If $\mathcal{D}$ is negative, then the eigenvalues of A_* from (76) are $\lambda_1 = 0$ and a conjugate pair and further studies have to be carried out.)

To see how we can satisfy Assumption 4, first consider the simplified dynamics $\dot{x} = Ax$ where A is the Jordan form of A_* and φ is not present, which corresponds to setting $a_1 = b_0 = 0$, and where $\mathcal{D} > 0$. Then λ_2 and λ_3 are negative, and (24) is the requirement $\beta = \exp(\rho A)$, by taking $\bar{V} = \bar{D} = 0$. Also, since we take $C = (1, 0, 0)$ in the output, the choices $Q = I$ and $K = (1/2, 0, 0)^\top$ give the diagonal matrix $\exp(\rho A^\top)(I - KC)^\top Q(I - KC)\exp(\rho A) = \text{diag}\{1/4, \exp(2\lambda_2 \rho), \exp(2\lambda_3 \rho)\}$. Hence, when $\varphi = 0$, we can satisfy the requirement

$$\exp(\rho A^\top)(I - KC)^\top Q(I - KC)\exp(\rho A) \preceq \kappa Q \qquad (77)$$

from Assumption 4 for all $\rho \in [\underline{\nu}, \bar{\nu}]$ with $\bar{V} = \bar{D} = 0$, $Q = I$, $K = (1/2, 0, 0)^\top$, any $\underline{\nu}$ and $\bar{\nu}$ satisfying $0 < \underline{\nu} < \bar{\nu}$, and

$$\kappa = \max\{1/4, \exp(2\lambda_2 \rho), \exp(2\lambda_3 \rho)\} \in (0, 1).$$

Therefore, by continuity of $\overline{\beta}$ and $\underline{\beta}$ in $\bar{D}$ and $\bar{V}$, Assumption 4 also holds for the motor dynamics $\dot{x} = Ax + \varphi(t, x)$ (with the same K, Q, $\underline{\nu}$, and $\bar{\nu}$ that we used in the $\varphi = 0$ case) when $a_1 > 0$ is a small enough constant and $\mathcal{D} \geq 0$, because we can make $\bar{D} > 0$ and $\bar{V} > 0$ as small as we want by making $a_1 > 0$ small enough (and because we can make $\rho > 0$ as small as we want to cover the case where $\lambda_2 = \lambda_3$, where there is a $\rho e^{\lambda_2 \rho}$ on the super-diagonal of $e^{A\rho}$).

We performed numerical tests to verify Assumption 4, using the linear matrix inequality condition from Proposition 1 with $\underline{\nu} = \bar{\nu}$. We chose $a_1 = 2$, $a_2 = 3$, $a_3 = 1$, $a_4 = 1$, $b_0 = 1$, and $b_1 = 1$. For these values, we can use the YALMIP package from Löfberg (2004) in MATLAB and the SeDuMi solver from Sturm (1999) to verify that (55) holds for all $\omega \in \mathcal{S}$ with $\nu = \underline{\nu} = \bar{\nu} = 0.4$. The correction term obtained in the initial coordinates is $K_* = P^{-1}K = (1,\ -0.196,\ -0.307)^\top$. We report our simulations in Figure 3 below, which show the good performance of our observer.

Remark 4 *There are many other possible choices of A_* and φ_*, besides the ones in (74). Another choice would be to replace $-a_1 \sin(x_{1*})$ in φ_* in (74) by the second order term $a_1(x_{1*} - \sin(x_{1*}))$, and then make the corresponding change in the second row of A_*. If the eigenvalues of A_* are all real numbers, then we can use more traditional time invariant changes of coordinates to produce real Jordan canonical forms that are also Metzler. The preceding analysis makes it possible to find explicit bounds on the allowable values of a_1 that can be used to satisfy our assumptions of our theorem for any choice of the stepsize ν.*

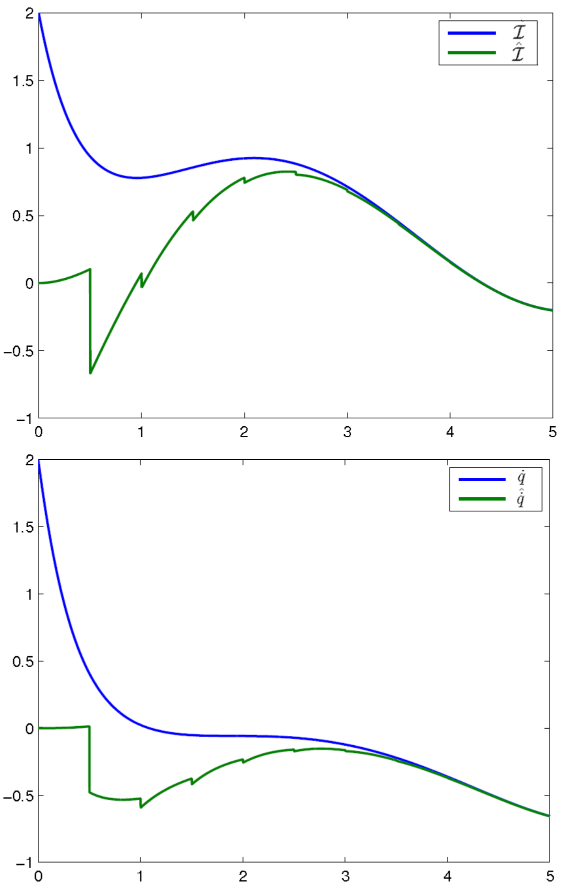

Fig. 3. Evolution with Time of the State Components $(x_2)_* = \mathcal{I}$ and $(x_3)_* = \dot{q}$ and their Estimation.

6 Conclusion

We gave a new method for building continuous-discrete observers for a broad class of continuous time systems where only sampled values of the output are available for measurement. We used a cooperative system approach that produces upper and lower bounds for fundamental solutions of time-varying systems. These bounds are of independent interest. We demonstrated our method using a pendulum dynamics, which showed how our approach gives a larger value for the maximal allowable measurement stepsize than the value that was available in the literature. We also illustrated our result using a motor dynamics, where a preliminary change of coordinates was used to put the system in a form that can be covered by our theorem. Some desirable extensions would quantify the effects of uncertainties in the discrete output observations, or time delays in the original system or in the observations, using input-to-state stability and Lyapunov-Krasovskii functionals.

Appendix

We used the following lemma at the end of our proof of Lemma 3 in Section 3.2:

Lemma A.1 *Let $K \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{n\times n}$ be any constant matrices, and set*

$$H = \begin{bmatrix} K & L \\ L & K \end{bmatrix} . \tag{A.1}$$

Let $\varsigma_1 : [0,\infty) \to \mathbb{R}^{2n\times 2n}$ and $\varsigma_2 : [0,\infty) \to \mathbb{R}^{2n\times 2n}$ be the functions

$$\varsigma_1(t) = e^{Ht} \quad \text{and} \quad \varsigma_2(t) = \frac{1}{2}\begin{bmatrix} J^\sharp & J^\flat \\ J^\flat & J^\sharp \end{bmatrix} , \tag{A.2}$$

where

$$J^\sharp = e^{(K+L)t} + e^{(K-L)t} \text{ and } J^\flat = e^{(K+L)t} - e^{(K-L)t}.$$

Then $\varsigma_1(t) = \varsigma_2(t)$ holds for all $t \geq 0$.

Proof: Simple calculations give

$$\dot{\varsigma}_2(t) = \frac{1}{2}\begin{bmatrix} L^\sharp & L^\flat \\ L^\flat & L^\sharp \end{bmatrix} , \tag{A.3}$$

where

$$L^\sharp = (K+L)e^{(K+L)t} + (K-L)e^{(K-L)t} \text{ and}$$
$$L^\flat = (K+L)e^{(K+L)t} - (K-L)e^{(K-L)t}.$$

By grouping terms, we can rewrite the right side of (A.3) as

$$\tfrac{1}{2}\begin{bmatrix} M^\sharp & M^\flat \\ M^\flat & M^\sharp \end{bmatrix},$$

where

$$M^\sharp = K\left(e^{(K+L)t} + e^{(K-L)t}\right) + L\left(e^{(K+L)t} - e^{(K-L)t}\right)$$

and

$$M^\flat = K\left(e^{(K+L)t} - e^{(K-L)t}\right) + L\left(e^{(K+L)t} + e^{(K-L)t}\right).$$

Therefore $\dot{\varsigma}_2(t) = H\varsigma_2(t)$ holds for all $t \geq 0$. Since $\dot{\varsigma}_1(t) = H\varsigma_1(t)$ holds for all $t \geq 0$ and $\varsigma_2(0) = \varsigma_1(0)$, we conclude that $\varsigma_2(t) = \varsigma_1(t)$ for all $t \geq 0$, by the uniqueness of solutions property. □

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