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***Abstract*—We propose a prediction based stabilization approach for a general class of nonlinear time-varying systems with pointwise delay in the input. It is based on a recent new prediction strategy, which makes it possible to circumvent the problem of constructing and estimating distributed terms in the expression for the stabilizing control laws. Our result applies in cases where other recent results do not.**

I. Introduction

Constructing delay tolerant feedback controls is a central problem that is of compelling ongoing research interest. Many important systems with input delays are time-varying. For instance, the problem of tracking trajectories leads to challenging problems for time-varying systems with input delays, as explained, e.g., in [23]. See also [6], [25], and [28] for surveys on delay compensating control.

There are several approaches to designing delay tolerant controls. One involves solving the problem with the input delay set to zero, and then searching for upper bounds on the input delays that the closed loop system can tolerate while still realizing the desired goal. This often involves Lyapunov-Krasovskii functionals (as discussed in [13] and [22]), which often lead to satisfactory results when the delay is small; see [21]. However, many applications have long input delays, such as communication networks or multi-agent problems where the agents are geographically dispersed. In general, stabilization under long input delays needs control designs that use the value of the input delay, and in many cases, distributed delays are used, meaning the control uses all values of the state or input along some interval of past times; see [4], [5], [24], and [29].

In this note, we revisit the problem of applying a prediction based approach to construct globally asymptotically stabilizing control laws for time-varying systems. We adapt the fundamental new approach of [26], which was limited to time-invariant linear systems that satisfy certain matrix inequalities. See also [18], where extensions to linear systems with time-varying delays and approximately known delays are provided. This approach differs from the classical reduction model approach or the prediction based approaches introduced by M. Krstic (as in [7], [15], and [17]) which also involve distributed terms.

We use several dynamic extensions, making it possible to obtain a prediction of the state variable without using distributed terms. Many contributions, including [2] and [3], use several dynamic extensions to carry out state prediction, but to the best of our knowledge, they do not apply to the problem we consider here and they use distributed terms. Our prediction stabilization technique applies to nonlinear Lipschitz systems, which is also the case for many prediction ones, e.g., [15]. We obtain an explicit result, which applies in cases where earlier results such as [23] do not apply. Our work [23] covered time-varying linear systems, using distributed terms, and included an application to a linearized pendulum tracking dynamics. To illustrate the value added by our new approach, we apply our new theorem to examples from [23], including a pendulum tracking control that does not use linearization and that does not require any distributed terms.

Prediction without distributed terms has been pursued in several significant works that do not address the challenges that we overcome here. For instance, [1], [8], [9], and [32] were limited to linear time-invariant systems $\dot{x} = Ax+Bu$, under addition eigenvalue conditions on A and controllability conditions or bounds on the delays, without robustness to uncertainty; [10] covers strict feedback systems under conditions on coefficient matrices in a new system that is obtained after a diffeomorphic transformation, which we do not require here; [11] used partial spectrum assignment to achieve prescribed decay rates for linear time-invariant systems; [12] was also confined to linear time-invariant systems; [14] covered nonlinear systems under a globally drift-observability condition that we also do not require; and [30] and [31] cover time-varying linear systems and give sufficient conditions for stabilizability under pseudo-predictor feedback using an integral delay system that is not needed here.

II. Definitions and Notation

Throughout the paper, we omit arguments of functions when they are clear, and the dimensions of the Euclidean spaces are arbitrary unless otherwise noted. The usual Euclidean norm of vectors, and the induced norm of matrices, are denoted by $|\cdot|$. We use $|\phi|_{\mathcal{I}}$ to denote the usual essential supremum of a function ϕ over any interval $\mathcal{I}$ contained in its domain. Consider any system of the form

$$\dot{y}(t) = \mathcal{G}(t, y(t), U(t-h), \mu(t)), \tag{1}$$

where the state y, the control U, and the unknown measurable locally essentially bounded perturbation μ are valued in any Euclidean spaces $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, and $\mathbb{R}^{n_3}$, respectively, and h is a constant delay. Due to the delay, its solutions are defined for given initial times $t_0 \geq 0$, initial functions that are defined on $[t_0-h, t_0]$, and choices of the function μ. We assume that (1) is forward complete, meaning all such solutions are uniquely defined on $[t_0-h, \infty)$; see below for assumptions that ensure the forward completeness property.

One desirable property for (1) is input-to-state stability (or ISS, which we also use to mean input-to-state stable); see, e.g., [16] for discussions on ISS for systems without delays. The ISS definition for delay systems can be expressed in terms of the standard classes $\mathcal{KL}$ and $\mathcal{K}_\infty$ of comparison functions, defined, e.g., in [16, Chapt. 4]. By ISS of (1), we mean that there are functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial times t_0, all initial functions, and all choices of the perturbations μ, the corresponding solutions of (1) all satisfy $|y(t)| \leq \beta(|y|_{[t_0-h,t_0]}, t-t_0) + \gamma(|\mu|_{[t_0,t]})$ for all $t \geq t_0$. This becomes the usual uniform global asymptotic stability property in the special case where there are no perturbations μ in (1). We use $\mathbb{N}$ to denote the natural numbers $\{1, 2, \ldots\}$. For any subsets S_1 and S_2 of Euclidean spaces, a function $W : S_1 \times S_2 \to \mathbb{R}^n$ is called locally Lipschitz in its second variable uniformly in its first variable provided for each constant $R > 0$, there is a constant $L_R > 0$ such that $|W(s_1, s_a) - W(s_1, s_b)| \leq L_R|s_a - s_b|$ holds for all $s_1 \in S_1$ and all s_a and s_b in S_2 for which $\max\{|s_a|, |s_b|\} \leq R$.

Key Words: Delay, robustness, stability, time-varying.

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III. Statement of Main Result

Consider the system

$$\dot{x}(t) = f(t, x(t), u(t-h), \delta(t)), \tag{2}$$

where the state x, the control u, and the unknown measurable locally essentially bounded perturbation δ are valued in $\mathbb{R}^n$, $\mathbb{R}^c$, and $\mathbb{R}^d$ respectively, and $h > 0$ is a constant delay. We design the control u to prove stability properties for this system when the delay h is known and arbitrarily large. We introduce these two assumptions, the first of which will be used to ensure forward completeness of (2):

Assumption 1: The function f is continuous, satisfies $f(t,0,0,0) = 0$ for all $t \geq 0$, and admits a constant $k > 0$ such that for all $t \geq 0$ and $U \in \mathbb{R}^c$, the inequality

$$|f(t,z_1,U,\Delta_1) - f(t,z_2,U,\Delta_2)| \leq k|z_1 - z_2| + k|\Delta_1 - \Delta_2| \tag{3}$$

holds for all $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^n$, $\Delta_1 \in \mathbb{R}^d$, and $\Delta_2 \in \mathbb{R}^d$. □

Assumption 2: There exists a continuous function $u_s : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^c$ that is locally Lipschitz in its second variable uniformly in its first variable such that the system

$$\dot{x}(t) = f(t, x(t), u_s(t, x(t) + \epsilon(t)), \delta(t)) \tag{4}$$

is ISS with respect to the combined disturbance (ε, δ). □

Before stating our theorem, we first convey the basic ideas of our method, which has two steps. First, we use a closed-loop system, which allows us to replace classical predictors with distributed terms by a new control that is based on a dynamic extension, and that exploits knowledge of future inputs using the delayed feedback property of the input. Second, since the delay h is finite, we use a chain of predictors to apply the method, to cover systems with arbitrarily long delays h. Letting $f_0(t,x,u) = f(t,x,u,0)$, we prove the following (but see Section V for additional assumptions that lead to a standard ISS estimate for our closed loop input delayed system):

Theorem 1: Let (2) satisfy Assumptions 1-2 and $m \in \mathbb{N}$ satisfy

$$m > 11.4kh. \tag{5}$$

Choose any constant $\lambda_a \in (0,1)$ such that

$$m > h\sqrt{(2/k)(1+\lambda_a)}(4k+\lambda_a)^{3/2}. \tag{6}$$

Then there are functions $\beta_d \in \mathcal{KL}$ and $\gamma_d \in \mathcal{K}_\infty$ such that all solutions $x(t)$ of the system (2), in closed loop with

$$u(t) = u_s(t+h, z_m(t)), \tag{7}$$

where z_m denotes the last n components of the state of the system

$$\left\{\begin{array}{rcl} \dot{z}_1(t) &=& f_0\left(t+\frac{h}{m}, z_1(t), \Phi(t,z_m,1)\right) \\ && -(4k+\lambda_a)[z_1\left(t-\frac{h}{m}\right) - x(t)] \\ \dot{z}_2(t) &=& f_0\left(t+\frac{2h}{m}, z_2(t), \Phi(t,z_m,2)\right) \\ && -(4k+\lambda_a)\left[z_2\left(t-\frac{h}{m}\right) - z_1(t)\right] \\ & \vdots & \\ \dot{z}_m(t) &=& f_0\left(t+h, z_m(t), \Phi(t,z_m,m)\right) \\ && -(4k+\lambda_a)\left[z_m\left(t-\frac{h}{m}\right) - z_{m-1}(t)\right] \end{array}\right. \tag{8}$$

where $\Phi(t,z_m,i) = u_s(t+h-h(m-i)/m, z_m(t-h(m-i)/m))$ for all $t \geq 0$ and $i \in \{1,2,\ldots,m\}$, satisfy

$$\begin{array}{l} |x(t)| \leq \\ \beta_d\left(|x|_{[t_0-h,t_0+\frac{h}{m}]} + |z|_{[t_0-h,t_0+\frac{h}{m}]}, t-t_0\right) + \gamma_d\left(|\delta|_{[t_0,t]}\right) \end{array} \tag{9}$$

for all $t_0 \geq \frac{h}{m}$ and all $t \geq t_0$. □

Remark 1: Notice that we can write $\Phi(t,z_m,i) = u(t-h(m-i)/m)$ for all t and i, where u is defined by (7). A key difference between Theorem 1 and our prior work [23] is that Theorem 1 provides a control with no distributed terms that applies to a broad class of time-varying nonlinear systems, while [23] is limited to time-varying linear systems and uses distributed terms. While [20] applies to more general nonlinear systems that are not necessarily globally Lipschitz, the controls in [20] use distributed terms and only ensure local stability, which is weaker than the global results we prove here.

Remark 2: Our condition $t_0 \geq h/m$ is used for our Lyapunov-Krasovskii analysis, and is purely technical. It can be relaxed, but for the sake of simplicity and space constraints, we do not modify it.

IV. Proof of Theorem 1

Throughout the proof, all equalities and inequalities are for all $t \geq 0$ and along all trajectories of the closed loop system, unless otherwise indicated. Set $p = 4k + \lambda_a$, $z_0 = x$, and $e_i(t) = z_i(t) - z_{i-1}(t + h/m)$ for all $i \in \{1,\ldots,m\}$. Our definition of f_0 gives

$$\begin{array}{l} \dot{e}_1(t) = -pe_1\left(t-\frac{h}{m}\right) \\ +f_0\left(t+\frac{h}{m}, z_1(t), u\left(t-\frac{h(m-1)}{m}\right)\right) \\ -f\left(t+\frac{h}{m}, x\left(t+\frac{h}{m}\right), u\left(t-\frac{h(m-1)}{m}\right), \delta\left(t+\frac{h}{m}\right)\right) . \end{array} \tag{10}$$

Similarly, since there are no disturbances δ in (8), we get

$$\begin{array}{l} \dot{e}_i(t) = -pe_i\left(t-\frac{h}{m}\right) \\ +f_0\left(t+i\frac{h}{m}, z_i(t), u\left(t-\frac{h(m-i)}{m}\right)\right) \\ -f_0\left(t+i\frac{h}{m}, z_{i-1}\left(t+\frac{h}{m}\right), u\left(t-\frac{h(m-i)}{m}\right)\right) + pe_{i-1}(t) \end{array} \tag{11}$$

when $i > 1$. We first analyze the e_1-subsystem (10).

The Fundamental Theorem of Calculus gives

$$\begin{array}{l} \dot{e}_1(t) = -pe_1(t) + p\int_{t-\frac{h}{m}}^{t} \dot{e}_1(\ell)\mathrm{d}\ell \\ +f_0\left(t+\frac{h}{m}, z_1(t), u\left(t-\frac{h(m-1)}{m}\right)\right) \\ -f\left(t+\frac{h}{m}, x\left(t+\frac{h}{m}\right), u\left(t-\frac{h(m-1)}{m}\right), \delta\left(t+\frac{h}{m}\right)\right) . \end{array} \tag{12}$$

Let $\nu(e_1) = \frac{1}{2}|e_1|^2$. Then Assumption 1 and our formula for f_0 give

$$\begin{array}{rcl} \dot{\nu}(t) &\leq& -p|e_1(t)|^2 + p\int_{t-\frac{h}{m}}^{t} e_1(t)^\top \dot{e}_1(\ell)\mathrm{d}\ell \\ && +|e_1(t)|k\left(\left|z_1(t) - x\left(t+\frac{h}{m}\right)\right| + \left|\delta\left(t+\frac{h}{m}\right)\right|\right) \\ &=& (k-p)|e_1(t)|^2 + p\int_{t-\frac{h}{m}}^{t} e_1(t)^\top \dot{e}_1(\ell)\mathrm{d}\ell \\ && +k|e_1(t)|\left|\delta\left(t+\frac{h}{m}\right)\right| . \end{array} \tag{13}$$

From Young's inequality, it follows that

$$\begin{array}{rcl} \dot{\nu}(t) &\leq& (k-p)|e_1(t)|^2 + k|e_1(t)|\left|\delta\left(t+\frac{h}{m}\right)\right| \\ && +p\int_{t-\frac{h}{m}}^{t}\left[\frac{m}{2h}|e_1(t)|^2 + \frac{h}{2m}|\dot{e}_1(\ell)|^2\right]\mathrm{d}\ell \\ &=& \left(k-\frac{1}{2}p\right)|e_1(t)|^2 + \frac{ph}{2m}\int_{t-\frac{h}{m}}^{t}|\dot{e}_1(\ell)|^2\mathrm{d}\ell \\ && +k|e_1(t)|\left|\delta\left(t+\frac{h}{m}\right)\right| . \end{array} \tag{14}$$

Next observe that (10) gives the following for all $\ell \geq 0$:

$$\begin{array}{l} |\dot{e}_1(\ell)| \leq p\left|e_1\left(\ell-\frac{h}{m}\right)\right| + \left|f_0\left(\ell+\frac{h}{m}, z_1(\ell), u\left(\ell-\frac{h(m-1)}{m}\right)\right)\right. \\ \left. -f\left(\ell+\frac{h}{m}, x\left(\ell+\frac{h}{m}\right), u\left(\ell-\frac{h(m-1)}{m}\right), \delta\left(\ell+\frac{h}{m}\right)\right)\right| \\ \leq p\left|e_1\left(\ell-\frac{h}{m}\right)\right| + k|e_1(\ell)| + k|\delta\left(\ell+\frac{h}{m}\right)|, \end{array}$$

by Assumption 1. Hence, Young's Inequality gives this for all $\ell \geq 0$:

$$\begin{array}{rcl} |\dot{e}_1(\ell)|^2 &\leq& 2p^2|e_1(\ell-h/m)|^2 + 2k^2\left(|e_1(\ell)|^2 + |\delta(\ell+h/m)|^2\right. \\ && \left.+2|e_1(\ell)||\delta(\ell+h/m)|\right) \\ &\leq& 2p^2|e_1(\ell-h/m)|^2 + 2k^2\left((1+\lambda_a/4)|e_1(\ell)|^2\right. \\ && \left.+(1+4/\lambda_a)|\delta(\ell+h/m)|^2\right). \end{array}$$

Therefore, (14) gives the following for all $t \geq \frac{h}{m}$:

$$\begin{aligned}\dot{\nu}(t) &\leq \left(k - \tfrac{1}{2}p\right)|e_1(t)|^2 + p^3 \tfrac{h}{m} \int_{t-2h/m}^{t-h/m} |e_1(\ell)|^2 \mathrm{d}\ell \\ &+ \tfrac{phk^2}{m}(1+\lambda_a/4) \int_{t-j/m}^{t} |e_1(\ell)|^2 \mathrm{d}\ell \\ &+ \tfrac{phk^2}{m}\left(1+\tfrac{4}{\lambda_a}\right) \int_{t}^{t+h/m} |\delta(\ell)|^2 \mathrm{d}\ell + k|e_1(t)||\delta(t+h/m)|.\end{aligned} \tag{15}$$

Since Young's Inequality also gives

$$k|e_1(t)||\delta(t+h/m)| \;\leq\; \tfrac{\lambda_a}{2}|e_1(t)|^2 + \tfrac{k^2}{2\lambda_a}|\delta(t+h/m)|^2, \tag{16}$$

we can use (16) to upper bound the last term in (15) and use our choice $p = 4k + \lambda_a$ and finally our choice of ν to get

$$\begin{aligned}\dot{\nu}(t) \;\leq\; & -k|e_1(t)|^2 + (4k+\lambda_a)^3 \tfrac{h}{m} \int_{t-2h/m}^{t-h/m} |e_1(\ell)|^2 \mathrm{d}\ell \\ & + (4k+\lambda_a)k^2(1+\lambda_a/4)\tfrac{h}{m} \int_{t-h/m}^{t} |e_1(\ell)|^2 \mathrm{d}\ell \\ & + \bar{M}|\delta|^2_{[t,t+h/m]} \\ \leq\; & -2k\nu(e_1(t)) + \tfrac{2h}{m}(4k+\lambda_a)^3 \int_{t-2h/m}^{t} \nu(e_1(\ell))\mathrm{d}\ell \\ & + \bar{M}|\delta|^2_{[t,t+h/m]}\,,\end{aligned} \tag{17}$$

where $\bar{M} = k^2/(2\lambda_a) + (phk^2/m)(1+4/\lambda_a)(h/m)$.

Let

$$\mu(e_{1,t}) = \nu(e_1(t)) + \tfrac{2h(1+\lambda_a)(4k+\lambda_a)^3}{m} \int_{t-2\frac{h}{m}}^{t} \int_{s}^{t} \nu(e_1(\ell))\mathrm{d}\ell \mathrm{d}s.$$

Here and in the sequel, $e_{i,t}$ is the restriction of e_i to $[t-2h/m, t]$ for all $t \geq t_0$ and $i \in \{1, 2, \ldots, m\}$. Then, for all $t \geq h/m$,

$$\begin{aligned}&\tfrac{d}{dt}\mu(e_{1,t}) \leq -2k\nu(e_1(t)) + \bar{M}|\delta|^2_{[t,t+h/m]} \\ &+ \tfrac{2h}{m}(4k+\lambda_a)^3 \left(\int_{t-2\frac{h}{m}}^{t} \nu(e_1(\ell))\mathrm{d}\ell + \tfrac{2h(1+\lambda_a)}{m}\nu(e_1(t))\right) \\ &- \tfrac{2h(1+\lambda_a)}{m}(4k+\lambda_a)^3 \int_{t-2\frac{h}{m}}^{t} \nu(e_1(\ell))\mathrm{d}\ell\,.\end{aligned} \tag{18}$$

Consequently,

$$\begin{aligned}&\tfrac{d}{dt}\mu(e_{1,t}) \leq 2k\left[-1 + \tfrac{2h^2}{m^2 k}(4k+\lambda_a)^3(1+\lambda_a)\right]\nu(e_1(t)) \\ &- \tfrac{2h}{m}\lambda_a(4k+\lambda_a)^3 \int_{t-2\frac{h}{m}}^{t} \nu(e_1(\ell))\mathrm{d}\ell + \bar{M}|\delta|^2_{[t,t+h/m]}.\end{aligned} \tag{19}$$

Therefore, our lower bound on m from (6) combined with

$$\mu(e_{1,t}) \leq \nu(e_1(t)) + \tfrac{4h^2}{m^2}(1+\lambda_a)(4k+\lambda_a)^3 \int_{t-2h/m}^{t} \nu(e_1(\ell))\mathrm{d}\ell$$

provides a constant $\epsilon_0 > 0$ such that, for all $t \geq h/m$,

$$\tfrac{d}{dt}\mu(e_{1,t}) \leq -\epsilon_0\left(\mu(e_{1,t}) + \int_{t-h/m}^{t} |e_1(\ell)|^2 \mathrm{d}\ell\right) + \bar{M}|\delta|^2_{[t,t+h/m]}$$

holds along all trajectories of the e_1 dynamics.

Similarly, since there are no δ's in the z system, (11) and the relation $2rs \leq \lambda_a r^2/4 + 4s^2/\lambda_a$ for all $r \geq 0$ and $s \geq 0$ give

$$\begin{aligned}|\dot{e}_i(\ell)|^2 \;\leq\; & 2p^2|e_i(\ell-h/m)|^2 + 2\left(k|e_i(\ell)| + p|e_{i-1}(\ell)|\right)^2 \\ \leq\; & 2p^2|e_i(\ell-h/m)|^2 + 2k^2(1+\lambda_a/4)|e_i(\ell)|^2 \\ & + 2p^2(1+4/\lambda_a)|e_{i-1}(\ell)|^2\end{aligned} \tag{20}$$

for any $i \in \{2, 3, \ldots, m\}$, which implies that the function

$$\begin{aligned}&\mu(e_{i,t}) = \\ &\tfrac{1}{2}|e_i(t)|^2 + \tfrac{2h}{m}(1+\lambda_a)(4k+\lambda_a)^3 \int_{t-2\frac{h}{m}}^{t} \int_{s}^{t} \nu(e_i(\ell))\mathrm{d}\ell \mathrm{d}s\end{aligned} \tag{21}$$

satisfies the following for all $t \geq h/m$:

$$\begin{aligned}\tfrac{d}{dt}\mu(e_{i,t}) \;\leq\; & -\epsilon_0\tilde{\mu}(e_{i,t}) + p|e_i(t)||e_{i-1}(t)| \\ & + \tfrac{p^3 h}{m}(1+4/\lambda_a) \int_{t-h/m}^{t} |e_{i-1}(\ell)|^2 \mathrm{d}\ell \\ \leq\; & -\epsilon_0\tilde{\mu}(e_{i,t}) + \tfrac{\epsilon_0}{4}|e_i(t)|^2 + \tfrac{p^2}{\epsilon_0}|e_{i-1}(t)|^2 \\ & + \tfrac{p^3 h}{m}(1+4/\lambda_a) \int_{t-h/m}^{t} |e_{i-1}(\ell)|^2 \mathrm{d}\ell \\ \leq\; & -\tfrac{\epsilon_0}{2}\tilde{\mu}(e_{i,t}) + \tfrac{p^2}{\epsilon_0}|e_{i-1}(t)|^2 \\ & + \tfrac{p^3 h}{m}(1+4/\lambda_a) \int_{t-h/m}^{t} |e_{i-1}(\ell)|^2 \mathrm{d}\ell,\end{aligned} \tag{22}$$

where

$$\tilde{\mu}(e_{i,t}) = \mu(e_{i,t}) + \int_{t-h/m}^{t} |e_i(\ell)|^2 \mathrm{d}\ell. \tag{23}$$

Therefore, the function

$$\mu^\sharp(e_t) = \sum_{i=1}^{m} \left(\frac{4}{\epsilon_0}\left(1 + \frac{p^2}{\epsilon_0} + \left(1+\frac{4}{\lambda_a}\right)\frac{p^3 h}{2m}\right)\right)^{m-i} \mu(e_{i,t})$$

where $e_t = (e_{1,t}, e_{2,t}, \ldots, e_{m,t})$ admits a constant $\bar{\epsilon} > 0$ such that

$$\begin{aligned}&\tfrac{d}{dt}\mu^\sharp(e_t) \leq -\bar{\epsilon}\mu^\sharp(e_t) \\ &+ \bar{M}\left(\tfrac{4}{\epsilon_0}\left(1 + \tfrac{p^2}{\epsilon_0} + \left(1+\tfrac{4}{\lambda_a}\right)\tfrac{p^3 h}{2m}\right)\right)^{m-1} |\delta|^2_{[t,t+h/m]}\end{aligned} \tag{24}$$

holds for all $t \geq h/m$ along all trajectories of the closed loop system. For instance, when $m > 2$, we get

$$\begin{aligned}&\tfrac{d}{dt}\left(\mu(e_{m,t}) + \tfrac{4}{\epsilon_0}\left(1 + \tfrac{p^2}{\epsilon_0} + \left(1+\tfrac{4}{\lambda_a}\right)\tfrac{p^3 h}{2m}\right)\mu(e_{m-1,t})\right) \\ &\leq -\tfrac{\epsilon_0}{2}\tilde{\mu}(e_{m,t}) + \Big\{\tfrac{p^2}{\epsilon_0}|e_{m-1}(t)|^2 \\ &\quad + \left(1+\tfrac{4}{\lambda_a}\right)\tfrac{p^3 h}{m} \int_{t-h/m}^{t} |e_{m-1}(\ell)|^2 \mathrm{d}\ell\Big\} \\ &\quad - \tfrac{\epsilon_0}{2}\tilde{\mu}(e_{m-1,t})\left(\tfrac{4}{\epsilon_0}\right)\left(1 + \tfrac{p^2}{\epsilon_0} + \left(1+\tfrac{4}{\lambda_a}\right)\tfrac{p^3 h}{2m}\right) \\ &\quad + \tfrac{4}{\epsilon_0}\left(1 + \tfrac{p^2}{\epsilon_0} + \left(1+\tfrac{4}{\lambda_a}\right)\tfrac{p^3 h}{2m}\right)\Big(\tfrac{p^2}{\epsilon_0}|e_{m-2}(t)|^2 \\ &\quad + \left(1+\tfrac{4}{\lambda_a}\right)\tfrac{p^3 h}{m} \int_{t-h/m}^{t} |e_{m-2}(\ell)|^2 \mathrm{d}\ell\Big) \\ &\leq -\tfrac{\epsilon_0}{2}\tilde{\mu}(e_{m,t}) - \tfrac{\epsilon_0}{2}\tilde{\mu}(e_{m-1,t})\left(\tfrac{4}{\epsilon_0}\right) \\ &\quad + \tfrac{4}{\epsilon_0}\left(1 + \tfrac{p^2}{\epsilon_0} + \left(1+\tfrac{4}{\lambda_a}\right)\tfrac{p^3 h}{2m}\right)\Big(\tfrac{p^2}{\epsilon_0}|e_{m-2}(t)|^2 \\ &\quad + \left(1+\tfrac{4}{\lambda_a}\right)\tfrac{p^3 h}{m} \int_{t-h/m}^{t} |e_{m-2}(\ell)|^2 \mathrm{d}\ell\Big)\,,\end{aligned} \tag{25}$$

by using the formula for $\tilde{\mu}(e_{m-1,t})$ to cancel the terms in curly braces in (25). Then (24) follows from a similar induction argument. Take any $t_0 \in [h/m, \infty)$. Using (24), an integrating factor, and the quadratic upper and lower bounds on ν, and setting $e = (e_1, e_2, \ldots, e_m)$, we can find $\beta_a \in \mathcal{KL}$ and $\gamma_a \in \mathcal{K}_\infty$ such that

$$|e(t)| \leq \beta_a(|e|_{[t_0-h,t_0]}, t - t_0 + h) + \gamma_a(|\delta|_{[t_0,t+h/m]}) \tag{26}$$

holds along all solutions of the closed loop system when $t_0 \geq h/m$.

Next observe that $z_m(t) = e_m(t) + z_{m-1}(t+h/m)$, $z_{m-1}(t) = e_{m-1}(t) + z_{m-2}(t+h/m)$, ..., and $z_1(t) = e_1(t) + x\left(t+\frac{h}{m}\right)$ all hold. Consequently, we can prove (e.g., by induction on m) that

$$z_m(t) = \sum_{l=0}^{m-1} e_{m-l}\left(t + l\frac{h}{m}\right) + x(t+h)\,. \tag{27}$$

Therefore, (2) in closed loop with the control (7) gives

$$\begin{aligned}&\dot{x}(t) = \\ &f\left(t, x(t), u_s\left(t, \sum_{l=0}^{m-1} e_{m-l}\left(t + l\frac{h}{m} - h\right) + x(t)\right), \delta(t)\right).\end{aligned} \tag{28}$$

From Assumption 2 and the fact that $\gamma(r+s) \leq \gamma(2r) + \gamma(2s)$ holds for all $r \geq 0$ and $s \geq 0$ and any $\gamma \in \mathcal{K}_\infty$, we can then find functions $\beta_b \in \mathcal{KL}$ and $\gamma_b \in \mathcal{K}_\infty$ such that for all $t \geq t_0$ and $t_0 \geq h/m$,

$$\begin{aligned}|x(t)| \;\leq\; & \beta_b(|x|_{[t_0-h,t_0]}, t - t_0) + \gamma_b(2m|e|_{[t-h,t-h/m]}) \\ & + \gamma_b(2|\delta|_{[t_0,t]}).\end{aligned} \tag{29}$$

Consider any $t \geq t_0$ and $s \in [t-h, t-h/m]$, and these two cases. (i) $s \in [t_0 - h, t_0]$. Then $|e(s)| \leq |e|_{[t_0-h,t_0]}$. (ii) $s \notin [t_0 - h, t_0]$. Then, $s \geq t_0$. From (26), we obtain

$$|e(s)| \leq \beta_a(|e|_{[t_0-h,t_0]}, s - t_0 + h) + \gamma_a(|\delta|_{[t_0,s+h/m]}). \tag{30}$$

We deduce that for all $s \in [t-h, t-h/m]$ the inequality

$$|e(s)| \leq \beta_c(|e|_{[t_0-h,t_0]}, s - t_0 + h) + \gamma_a(|\delta|_{[t_0,s+h/m]}) \tag{31}$$

holds with $\beta_c(q,r) = \beta_a(q,r) + qe^{-r+h}$. This gives $|e|_{[t-h,t-h/m]} \le \beta_c(|e|_{[t_0-h,t_0]}, t-t_0) + \gamma_a(|\delta|_{[t_0,t]})$. This inequality in combination with (29) gives

$$\begin{aligned} |x(t)| &\le \beta_b(|x|_{[t_0-h,t_0]}, t-t_0) \\ &\quad +\gamma_b\left(2m\beta_c(|e|_{[t_0-h,t_0]}, t-t_0)\right. \\ &\quad \left.+2m\gamma_a(|\delta|_{[t_0,t]})\right) + \gamma_b(2|\delta|_{[t_0,t]}) \\ &\le \beta_b(|x|_{[t_0-h,t_0]}, t-t_0) \\ &\quad +\gamma_b\left(4m\beta_c(|e|_{[t_0-h,t_0]}, t-t_0)\right) \\ &\quad +\gamma_b\left(4m\gamma_a(|\delta|_{[t_0,t]})\right) + \gamma_b(2|\delta|_{[t_0,t]}). \end{aligned} \tag{32}$$

Also, our formulas for the e_i's give $|e|_{[t_0-h,t_0]} \le 2m|z|_{[t_0-h,t_0+h/m]} + |x|_{[t_0-h,t_0+h/m]}$, where $z = (z_1, z_2, \dots, z_m)$. Hence, we can choose $\beta_d(q,r) = \beta_b(q,r) + \gamma_b(4m\beta_c(2mq,r))$ and $\gamma_d(r) = \gamma_b(4m\gamma_a(r)) + \gamma_b(2r)$.

V. Remarks on Assumptions and ISS

The conclusion of Theorem 1 is weaker than ISS because the right side of (9) involves x and z values on $[t_0-h, t_0+h/m]$. We can strengthen its conclusion to an ISS estimate provided we (a) fix any function $\bar{\alpha} \in \mathcal{K}_\infty$ and only consider initial functions that satisfy $|z_i|_{[t_0-h,t_0]} \le \bar{\alpha}(|x|_{[t_0-h,t_0]})$ for all $i \in \{1,2,\dots,m\}$ and (b) assume that there are constants $\bar{L}_1 > 0$ and $\bar{L}_2 > 0$ such that

$$|f(t,x,U,0)| \le \bar{L}_1(|x|+|U|) \ \text{ and } \ |u_s(t,x)| \le \bar{L}_2|x| \tag{33}$$

hold for all $t \ge 0$, $x \in \mathbb{R}^n$, and $U \in \mathbb{R}^c$, as follows.

Assumption 1 combined with the first bound in (33) implies that $|f(t,x,U,\Delta)| \le \bar{L}_1(|x|+|U|) + k|\Delta|$ holds for all $t \ge 0$, $x \in \mathbb{R}^n$, $U \in \mathbb{R}^c$, and $\Delta \in \mathbb{R}^d$. Hence, our choice of the control in Theorem 1 implies that along all trajectories of the closed loop system, we get

$$\begin{aligned} |\dot{x}(t)| &\le \bar{L}_1\big(|x(t)| + \bar{L}_2|z_m(t-h)|\big) + k|\delta(t)| \ \text{ and} \\ |\dot{z}_i(t)| &\le \bar{L}_1|z_i(t)| + (4k+\lambda_a)\left(\left|z_i\left(t-\tfrac{h}{m}\right)\right| + |z_{i-1}(t)|\right) \\ &\quad +\bar{L}_1\bar{L}_2\left|z_m\left(t-\tfrac{h(m-i)}{m}\right)\right| \end{aligned}$$

for all $t \in [t_0, t_0+h/m]$ and $i \in \{1,2,\dots,m\}$. Hence,

$$\begin{aligned} |(x(t),z(t))| &\le (m+1)|(x(t_0),z(t_0))| + \bar{L}_1\textstyle\int_{t_0}^t |x(\ell)|\mathrm{d}\ell \\ &\quad + \tfrac{h}{m}\bar{L}_1\bar{L}_2|z_m|_{[t_0-h,t_0]} + \tfrac{hk}{m}|\delta|_{[t_0,t]} \\ &\quad + m\bar{L}_1\textstyle\int_{t_0}^t |z(\ell)|\mathrm{d}\ell \\ &\quad + m\bar{L}_1\bar{L}_2\left(\tfrac{h}{m}|z_m|_{[t_0-h,t_0]} + \textstyle\int_{t_0}^t |z_m(\ell)|\mathrm{d}\ell\right) \\ &\quad + m(4k+\lambda_a)\left(|z|_{[t_0-h,t_0]} + \textstyle\int_{t_0}^t |z(\ell)|\mathrm{d}\ell\right) \\ &\le \bar{L}_3|(x,z)|_{[t_0-h,t_0]} + \bar{L}_4\textstyle\int_{t_0}^t |(x(\ell),z(\ell))|\mathrm{d}\ell \\ &\quad + \tfrac{hk}{m}|\delta|_{[t_0,t]} \end{aligned}$$

for all $t \in [t_0, t_0+h/m]$, where $\bar{L}_3 = m+1+(h/m)\bar{L}_1\bar{L}_2(1+m) + (4k+\lambda_a)m$ and $\bar{L}_4 = \bar{L}_1(1+m(1+\bar{L}_2)) + m(4k+\lambda_a)$. Hence, Gronwall's Inequality gives

$$\begin{aligned} &|(x(t),z(t))| \le \left(\bar{L}_3|(x,z)|_{[t_0-h,t_0]} + \tfrac{hk}{m}|\delta|_{[t_0,t]}\right)e^{\bar{L}_4 h/m} \\ &\le \left(\bar{L}_3|x|_{[t_0-h,t_0]} + m\bar{L}_3\bar{\alpha}\left(|x|_{[t_0-h,t_0]}\right) + \tfrac{hk}{m}|\delta|_{[t_0,t]}\right)e^{\bar{L}_4 h/m} \end{aligned}$$

for all $t \in [t_0-h, t_0+h/m]$. Therefore, in terms of the functions $\beta_d \in \mathcal{KL}$ and $\gamma_d \in \mathcal{K}_\infty$ from our proof of Theorem 1, the choices

$$\begin{aligned} \beta_e(q,r) &= \beta_d\left(4\bar{L}_3(q+m\bar{\alpha}(q))e^{\bar{L}_4 h/m}, r\right) \ \text{ and} \\ \gamma_e(r) &= \beta_d\left(\tfrac{4hk}{m}e^{\bar{L}_4 h/m}r, 0\right) + \gamma_d(r) \end{aligned} \tag{34}$$

are of class $\mathcal{KL}$ and $\mathcal{K}_\infty$ respectively and give the final ISS estimate $|x(t)| \le \beta_e\left(|x|_{[t_0-h,t_0]}, t-t_0\right) + \gamma_e(|\delta|_{[t_0,t]})$ for all $t \ge t_0$, using the relation $\max\{|x(t)|, |z(t)|\} \le |(x(t),z(t))|$.

VI. Examples and Simulations

We illustrate the value added by Theorem 1 by revisiting two examples from [23]. Our treatment of the examples differs from [23] because (i) the controls from [23] have distributed terms and (ii) the work [23] only applied to linear time-varying systems and so only applied to a linearized pendulum, unlike our treatment below that covers the time-varying nonlinear pendulum dynamics.

A. Scalar Example

The one dimensional example from [23, Section VII.A] is

$$\dot{x}(t) = \big[\sin(t/9) + \sin(11\pi t)\big]x(t) + u(t-h) + \delta(t) \tag{35}$$

with the disturbance δ. It is beyond the scope of standard reduction model technique or prediction results since the vector fields are time-varying and not periodic. However, [23, Theorem 1] provides functions $\bar{\beta} \in \mathcal{KL}$ and $\bar{\gamma} \in \mathcal{K}_\infty$ such that along all trajectories of (35), in closed loop with the control u that is defined by

$$u(t) = K(t)\left[x(t) + \textstyle\int_{t-h}^t \lambda(t, r+h)u(r)\mathrm{d}r\right], \tag{36}$$

with the choices

$$\begin{aligned} &\lambda(t,\ell) = \\ &e^{-9\cos(t/9) - \frac{1}{11\pi}\cos(11\pi t) + 9\cos(\ell/9) + \frac{1}{11\pi}\cos(11\pi\ell)} \ \text{ and} \\ &K(t) = \\ &e^{9\cos(t/9) + \frac{1}{11\pi}\cos(11\pi t) - 9\cos\left(\frac{t+h}{9}\right) - \frac{1}{11\pi}\cos(11\pi(t+h))} \\ &\times\big[-1 - \sin(t/9) - \sin(11\pi t)\big], \end{aligned} \tag{37}$$

and for all choices of the constant delay $h > 0$, we have

$$\begin{aligned} &|x(t)| + |u|_{[t-h,t]} \le \\ &\overline{\beta}\big(|x(t_0)| + |u|_{[t_0-h,t_0]}, t-t_0\big) + \overline{\gamma}(|\delta|_{[t_0,t]}) \end{aligned} \tag{38}$$

for all initial times $t_0 \ge 0$ and all $t \ge t_0$. By contrast, Theorem 1 above provides a control without distributed terms, as follows.

The assumptions of Theorem 1 are satisfied with the choices $k = 2$, $u_s(t,x) = -[1 + \sin(t/9) + \sin(11\pi t)]x$, $h = 1$, $m = 23$ and $\lambda_a = 0.0225$, and without any restrictions on the sup norm of the disturbance or on the initial functions, and the corresponding control (7) has no distributed terms. We undertook three simulations of the closed loop system (35) with the preceding choices and $t_0 = 0$, using `NDSolve` in the program Mathematica; see [19] and Remark 2. In our first two simulations, we chose the initial functions for x and the z_i's all equaling 1 on $[-1, 0]$. We chose the zero perturbation $\delta(t) = 0$ in the first simulation, and our second simulation used $\delta(t) = e^{-t}\sin(t)$. In our third simulation, we chose the same values as in our first two simulations, except with all of the initial functions equaling 7 on $[-1, 0]$, instead of 1, and with the perturbation $\delta(t) = 0.1\sin(t)$. We plotted the corresponding simulations for $x(t)$ in Fig. 1 below. They illustrate convergence towards 0, with overshoots depending on the perturbation values, and therefore help illustrate Theorem 1.

B. Pendulum

We next illustrate our theory using the model

$$\left\{\begin{array}{rcl} \dot{r}_1(t) &=& r_2(t) \\ \dot{r}_2(t) &=& -\frac{g}{l}\sin(r_1(t)) + \frac{1}{\mathcal{M}l^2}v(t-h) \end{array}\right. \tag{39}$$

of the simple pendulum, where $g = 9.8$ m/s is the gravity constant, l is the pendulum length in meters, $\mathcal{M}$ is the pendulum mass, v is the input, and h is the constant delay. As in [23], we wish to track a C^1 reference trajectory $(r_{1,s}(t), r_{2,s}(t))$ such that $\dot{r}_{1,s}(t) = r_{2,s}(t)$. The error variables $\tilde{r}_i = r_i - r_{i,s}(t)$ for $i = 1,2$ and the feedback

$$u(t-h) = \tfrac{1}{\mathcal{M}l^2}v(t-h) - \dot{r}_{2,s}(t) - \tfrac{g}{l}\sin(r_{1,s}(t)) \tag{40}$$

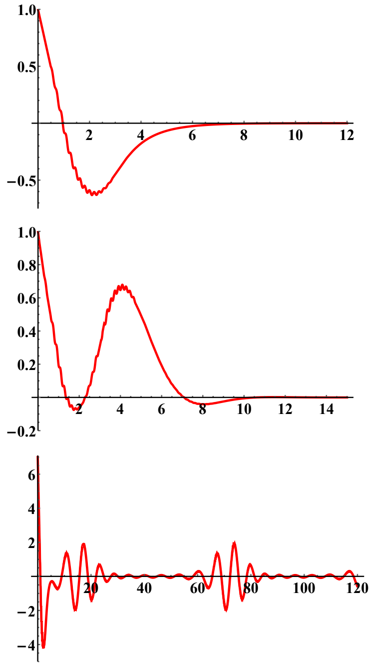

Fig. 1. Simulating $x(t)$ for (35) with $h = 1$ using Control (7) with $m = 23$, $k = 2$, $\lambda_a = 0.0225$, and $x(r) = z_i(r) = \bar{x}$ for all i and $r \in [-1, 0]$, and Different Perturbations δ. Top: $\delta(t) = 0$ with $\bar{x} = 1$. Middle: $\delta(t) = e^{-t}\sin(t)$ with $\bar{x} = 1$. Bottom: $\delta(t) = 0.1\sin(t)$ with $\bar{x} = 7$.

produce the tracking dynamics

$$\left\{\begin{array}{rcl} \dot{\tilde{r}}_1(t) & = & \tilde{r}_2(t) \\ \dot{\tilde{r}}_2(t) & = & \frac{g}{l}[\sin(r_{1,s}(t)) - \sin(\tilde{r}_1(t) + r_{1,s}(t))] \\ & & + u(t-h). \end{array}\right. \tag{41}$$

The work [23] showed that for the case where $r_{1,s}(t) = \omega t$ where $\omega > 0$ is a large enough constant and $h = 1$, the linearization

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & -\frac{g}{l}\cos(\omega t)x_1(t) + u(t-h), \end{array}\right. \tag{42}$$

of (41) at 0 has the globally exponentially stabilizing distributed control u defined by

$$\begin{array}{l} u(t) = -0.6x_1(t) - 0.4x_2(t) \\ -\int_{t-1}^{t} \big(0.6(t-r-1) + 0.4\big)u(r)\mathrm{d}r. \end{array}$$

However, [23] does not cover the original nonlinear pendulum dynamics (41) and it imposes conditions on ω, and [23] cannot be applied to (42) without imposing an upper bound on h. By contrast, (41) is covered by Theorem 1 without any bounds on the delay or on ω, and our theorem provides a controller with no distributed terms.

To apply Theorem 1 to (41), we chose $\mathcal{M} = h = 1$, $l = 10$,

$$u_s(t, x) = -0.98(\sin(\omega t) - \sin(x_1 + \omega t)) - x_1 - x_2, \tag{43}$$

and $x = (x_1, x_2) = (\tilde{r}_1, \tilde{r}_2)$. For simplicity, we only consider the case where $\omega = 1$, but analogous reasoning applies for any positive constants ω, h, $\mathcal{M}$, and l. The requirements of our theorem are then satisfied with $k = 1$, $m = 16$, and $\lambda_a = 0.35$. We simulated the corresponding perturbed pendulum dynamics

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & \frac{g}{l}[\sin(t) - \sin(x_1(t) + t)] + u(t-h) + \delta(t) \end{array}\right. \tag{44}$$

with the control from our theorem, again using the command `NDSolve` in the Mathematica program, with different choices of δ, which represents control uncertainty. Recalling our change of feedback with $u = u_s$, it follows that the original controller becomes

$$v(t) = 98\sin(z_{m1}(t) + t + 1) - 100(z_{m1}(t) + z_{m2}(t)), \tag{45}$$

where $z_m = (z_{m1}, z_{m2})$ is the bottom system in the dynamic control corresponding to $m = 16$. In our first simulation, we took the initial functions for all components of x and the z_i's from our dynamic controller to be identically 1, again with the initial time $t_0 = 0$, and with the zero perturbation $\delta = 0$. In Fig. 2, we plotted the corresponding paths for the tracking error components $x_1(t)$ and $x_2(t)$ and the control $v(t-1)$. The plots show rapid convergence of the tracking error towards 0, and the convergence of the control values, and therefore help demonstrate our theory. In Fig. 3, we show the solutions from the first simulation in red, and in blue we show the corresponding uncontrolled trajectories for x_1 and x_2 that we obtained by simulating (44) with $u = \delta = 0$ in (44), for comparison. When $u = 0$, the solution $x_1(t)$ obtained from Mathematica for the unperturbed $\delta = 0$ case satisfies $\lim_{t\to\infty} x_1(t) = \infty$, which is another motivation for our new control design.

Then we repeated the simulations with the same choices as our first pendulum simulations, except with the initial functions for all components of x and the z_i's from our control changed to be identically 1.5, and with $\delta(t) = 0.05\sin(t)$. In Fig. 4, we show the paths for the tracking error components $x_1(t)$ and $x_2(t)$ and $v(t-1)$ in this case. With the perturbation, the tracking errors no longer converge to zero. However, they oscillate around 0, which is consistent with Theorem 1, so this too demonstrates our theory.

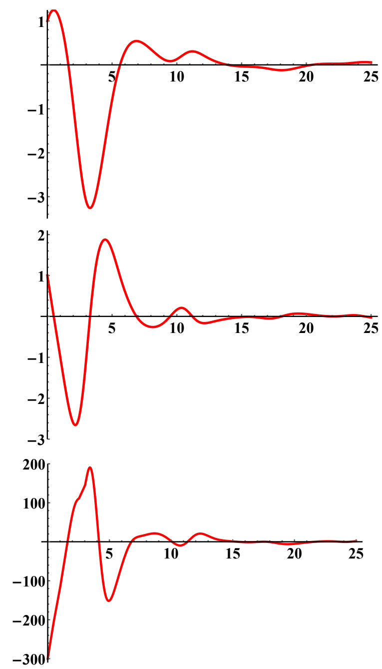

Fig. 2. Simulation of Nonlinear Pendulum Tracking System (44) with all Components of Initial Functions for x and the z_i's set to 1 on $[-1, 0]$ with Perturbation $\delta(t) = 0$. Top to Bottom: $x_1(t)$, $x_2(t)$, and $v(t-1)$ from (45).

VII. Conclusions

Stabilization under arbitrarily long feedback delays leads to controllers derived from reduction model or prediction techniques, which can be difficult to implement because of the need to store past control values or to introduce possibly unstable dynamic extensions and evaluate the distributed terms in practice. We proposed an alternative stabilization technique for nonlinear systems with an arbitrarily long input delay. Potential advantages of our method include (a) that it produces a predictive state estimate that can be used in the nominal

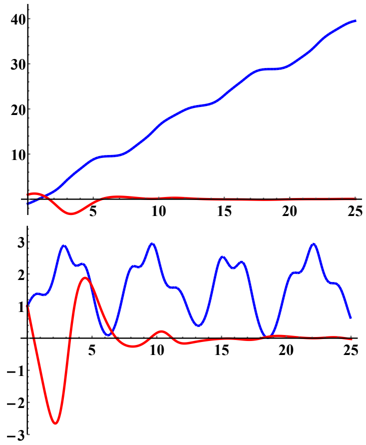

Fig. 3. Simulation of (44) from Fig. 2 (red) Showing Convergence, with Corresponding Simulations of Uncontrolled System Obtained by Setting $u = \delta = 0$ in (44) (blue) Showing Divergence. Top to Bottom: $x_1(t)$ and $x_2(t)$.

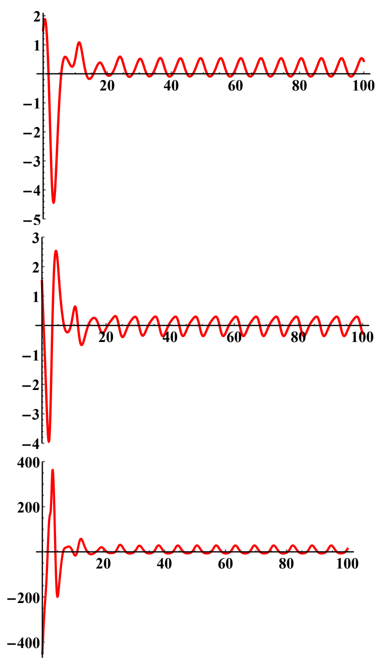

Fig. 4. Simulation of Nonlinear Pendulum Tracking System (44) with all Components of Initial Functions for x and the z_i's set to 1.5 with $\delta(t) = 0.05\sin(t)$. Top to Bottom: $x_1(t)$, $x_2(t)$, and $v(t-1)$ from (45).

control for the corresponding undelayed system, so our control can satisfy input constraints when the constraints are satisfied by the nominal control, (b) that our control does not involve distributed terms, and (c) that the closed-loop systems enjoy an ISS robustness property. We hope to generalize our work to systems with sampled data, as in [27], using state values at aperiodic discrete times.

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