# Differentiable Lyapunov Function and Center Manifold Theory

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**Abstract.** Continuously differentiable Lyapunov functions for nonlinear systems whose the asymptotic stability can be proved via the center manifold theory are constructed. They are of particular interest when some parameters of the systems are not exactly known.

## 1 Introduction

The objective of the center manifold theory is the analysis of the local stability or the stabilizability properties of systems which admit a linear approximation which is not globally asymptotically stabilizable. This theory, exposed in [2] and in [1], has proved his great usefulness in various fields of the nonlinear control theory. As a direct consequence of this theory, there is for instance the theory of singular perturbations, and some designs of stabilizing control laws for some nonholonomic systems (see [5]).

As explained in [1], one of the great advantages of the center manifold theory is that it is entirely independent of one's ability to construct a Lyapunov function, which is a priori not an easy task. However, it is well-known that the knowledge of a Lyapunov function can be of great help when problems such as, for instance, robustness issues or the determination of a subset of the basin of attraction of an asymptotically stable system must be addressed. This consideration motivates the purpose of the present note which is to establish a link between the two major parts of the theory of the stability of nonlinear systems that are the center manifold theory and the Lyapunov approach.

The proof of the centre manifold theorem given in [3] is based on the construction of a Lyapunov function. Unfortunately, the proposed Lyapunov function is not continuously differentiable. This lack of smoothness is a drawback for several reasons:

• The Lyapunov function cannot be used as a tool in a backstepping or a forwarding context (these approaches require the knowledge of at least $C^1$ Lyapunov functions for some subsystems).

• When some parameters of the system are not exactly known, the Lyapunov function cannot be constructed (see our example in Section 3).

• Significant robustness properties cannot be inferred from it.

The main result of our work consists of showing how by slightly modifying the Lyapunov function provided in [3] a smooth Lyapunov function is obtained. We borrow from [4] the technique of proof we adopt.

The paper is organized as follows. In Section 2, the main result is stated and proved. An example in Section 3 illustrates the usefulness of our Lyapunov construction in the context of systems with parameters not exactly known. Some concluding remarks in Section 4 end the paper.

**Preliminaries.**
1. For a real valued $C^1$ function $k(\cdot)$, we denote by $k'(\cdot)$ its first derivative.
2. We assume throughout the paper that the functions encountered are sufficiently smooth.
3. A function $\gamma(X)$ is of order one (resp. two) at the origin if for some $c > 0$, the inequality $|\gamma(X)| \leq c|X|$ (resp. $|\gamma(X)| \leq c|X|^2$) is satisfied on a neighborhood of the origin.
4. A function $V(\cdot)$ on $\Re^n$ is positive definite if $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$.
5. A positive definite function $V(\cdot)$ on $\Re^n$ is a strict Lyapunov function for the system

$$\dot{\chi} = \varphi(\chi) \tag{1}$$

if

$$\frac{\partial V}{\partial \chi}(\chi)\varphi(\chi) < 0 \, , \, \forall \chi \neq 0 \tag{2}$$

## 2 Main result

Consider the nonlinear system

$$\begin{cases} \dot{x} = Mx + \alpha(x,z) \\ \dot{z} = f(x,z) \end{cases} \tag{3}$$

where $x \in \Re^{n_x}$, $z \in \Re^{n_z}$ are the components of the state, and introduce a set of assumptions.
**Assumption A1.** The matrix $M$ is Hurwitz and $\frac{\partial f}{\partial z}(0,0)$ is a critically stable matrix i.e. all the eigenvalues of $\frac{\partial f}{\partial z}(0,0)$ are on the imaginary axis. Moreover $\alpha(x,z)$ is a function of order two at the origin.

**Assumption A2.** There exists a function $h(z)$ of order two at the origin such that

$$\frac{\partial h}{\partial z}(z)f(h(z),z) = Mh(z) + \alpha(h(z),z) \tag{4}$$

**Assumption A3.** Two positive definite functions $V(\cdot)$, $W(\cdot)$ such that on a neighborhood of the origin

$$\frac{\partial V}{\partial z}(z)f(h(z),z) \leq -W(z) \tag{5}$$

are known.

**Remark 1.** According to Assumption A1, one can determine two positive definite symmetric matrices $Q$ and $R$ such that

$$QM + M^{\top}Q = -R \tag{6}$$

**Remark 2.** When the Assumptions A1 to A3 are satisfied, one can prove the local asymptotic stability of the system (3) by invoking the center manifold theory.

Let us state the main result.

**Theorem 1.** *Assume that the system (3) satisfies the Assumptions A1 to A3. Then there exists a strictly a continuously differentiable function $l(\cdot)$ zero at zero with a strictly positive first derivative such that the function*

$$U(x,z) = l(V(z)) + (x-h(z))^{\top}Q(x-h(z)) \tag{7}$$

*is a strict Lyapunov function for the system (3).*

**Proof.** Let us introcuce a new variable $\xi = x - h(z)$. Its time derivative is:

$$\begin{aligned}\dot{\xi} &= Mx + \alpha(x,z) - \tfrac{\partial h}{\partial z}(z)f(x,z)\\ &= M\xi + \alpha(h(z),z) - \tfrac{\partial h}{\partial z}(z)f(h(z),z) + Mh(z)\\ &\quad + \alpha(x,z) - \alpha(h(z),z) + \tfrac{\partial h}{\partial z}(z)f(h(z),z) - \tfrac{\partial h}{\partial z}(z)f(x,z)\\ &= M\xi + \alpha(x,z) - \alpha(h(z),z) + \tfrac{\partial h}{\partial z}(z)\left[f(h(z),z) - f(x,z)\right]\end{aligned} \tag{8}$$

Since the functions $\alpha(x,z)$ and $h(z)$ are of order two at the origin, we deduce that there exists a function $r(z,\xi)$ of order one such that

$$\dot{\xi} = M\xi + r(z,\xi)\xi \tag{9}$$

On the other hand, there exists a function $g(z,\xi)$ such that

$$\dot{z} = f(h(z),z) + g(z,\xi)\xi \tag{10}$$

We construct now a Lyapunov function for the system (3) using the representation (9)(10). One can check readily that the derivative of the function (7) along the solutions of (9)(10) satisfies:

$$\begin{aligned}\dot{U}(x,z) \leq\ & -l'(V(z))W(z) + l'(V(z))\tfrac{\partial V}{\partial z}(z)g(z,\xi)\xi - \xi^{\top}R\xi\\ & + 2\xi^{\top}Qr(z,\xi)\xi\end{aligned} \tag{11}$$

Since the function $r(z,\xi)$ is of order one, on a sufficiently small neighborhood of the origin the inequality

$$2\xi^\top Q r(z,\xi)\xi \ \leq \ \frac{1}{4}\xi^\top R\xi \tag{12}$$

holds. On the other hand, when $a \geq 0$ is sufficently large constant,

$$\begin{array}{rl} l'(V(z))\frac{\partial V}{\partial z}(z)g(z,\xi)\xi \leq \frac{1}{4a}\xi^2 + a\left[l'(V(z))\frac{\partial V}{\partial z}(z)g(z,\xi)\right]^2 \\ \leq \frac{1}{4}\xi^\top R\xi + a\left[l'(V(z))\frac{\partial V}{\partial z}(z)g(z,\xi)\right]^2 \end{array} \tag{13}$$

Combining (11), (12), (13), we obtain

$$\dot{U}(x,z) \ \leq \ -l'(V(z))W(z) + a\left[l'(V(z))\frac{\partial V}{\partial z}(z)g(z,\xi)\right]^2 - \frac{1}{2}\xi^\top R\xi \tag{14}$$

Since $W(z)$ and $V(z)$ are positive definite functions, one can determine continuously differentiable strictly increasing functions zero at zero $\alpha_i(\cdot)$, $i = 1$ to 3 such that:

$$W(z) \ \geq \ \alpha_1(|z|)\ , \ \left|\frac{\partial V}{\partial z}(z)\right| \ \leq \ \alpha_2(|z|)\ , \ V(z) \ \leq \ \alpha_3(|z|) \tag{15}$$

According to (14) and (15) the inequality

$$\dot{U}(x,z) \ \leq \ -\frac{1}{2}l'(V(z))W(z) - \frac{1}{2}\xi^\top R\xi \tag{16}$$

is satisfied if:

$$al'(\alpha_3(|z|))\alpha_2(|z|)^2|g(z,\xi)|^2 \ \leq \ \frac{1}{2}\alpha_1(|z|) \tag{17}$$

On any neighborhood of the origin, there exists $\Gamma > 0$ such that $|g(z,\xi)|^2 \leq \Gamma$. As a consequence, the previous inequality holds if, on a neighborhood of the origin,

$$l'(s) \ \leq \ \frac{1}{2a\Gamma}\frac{\alpha_1(\alpha_3^{-1}(s))}{\alpha_2(\alpha_3^{-1}(s))^2} \tag{18}$$

Since the functions $\alpha_i(\cdot)$'s are zero at zero and stricly increasing and continuously differentiable, a function $l(\cdot)$ strictly increasing, zero at the zero, continuously differentiable and such that the previous inequality is satisfied can be easily determined.

This concludes the proof.

## 3 Example

In this section, we illustrate on a very simple example how Theorem 1 can be used to construct a Lyapunov function when some parameters are not exactly known.

Consider the following two-dimensional system [3, Example 4.15]

$$\begin{aligned} \dot{x} &= -x + az^2 \\ \dot{z} &= zx \end{aligned} \tag{19}$$

Our objective is the construction of a smooth Lyapunov function in the case where $a$ is approximately known. We suppose that $a = b + \overline{\varepsilon}$ with $\overline{\varepsilon} \in [-\varepsilon, \varepsilon]$ and that $b < 0$ and $\varepsilon$ are known. To simplify, let $b = -1$.

In [3, Example 4.15], it is shown that the solution of the center manifold equation

$$h'(z)[zh(z)] + h(z) - z^2 = 0 \quad h(0) = h'(0) = 0 \tag{20}$$

is $h(z) = z^2 + O(|z|^3)$. This expression of $h(z)$ leads us to perform the change of coordinate $\xi = x + z^2$, which transforms (19) into

$$\begin{aligned} \dot{\xi} &= -\xi + \overline{\varepsilon} z^2 + 2z^2\xi - 2z^4 \\ \dot{z} &= -z^3 + z\xi \end{aligned} \tag{21}$$

From the proof of [3, Theorem 4.15] we deduce that the derivative of function

$$\nu(z, \xi) = 2z^2 + |\xi| \tag{22}$$

along the solutions of (21) is negative definite when $\overline{\varepsilon} = 0$. But one can check that it is not so when $\overline{\varepsilon} \neq 0$.

By applying Theorem 1, one can prove that the function

$$U(z, \xi) = \frac{1}{2}\xi^2 + \frac{1}{2}z^2 \tag{23}$$

is a strict Lyapunov function for (19), when $\varepsilon$ is smaller than a constant we will determine. The derivative of $U(z, \xi)$ along the solutions of (21) satisfies

$$\begin{aligned} \dot{U}(z, \xi) &= -\xi^2 - z^4 + z^2\xi + \overline{\varepsilon}\xi z^2 + 2z^2\xi^2 - 2\xi z^4 \\ &\leq -\left(\tfrac{1}{2} - \tfrac{\varepsilon}{2}\right)\xi^2 - \left(\tfrac{1}{2} - \tfrac{\varepsilon}{2}\right)z^4 + 2z^2\xi^2 - 2\xi z^4 \end{aligned} \tag{24}$$

It follows that when $\varepsilon \leq \frac{1}{2}$ then

$$\dot{U}(z, \xi) \leq -\tfrac{1}{8}\xi^2 - \tfrac{1}{8}z^4 \tag{25}$$

on the neighborhood of the origin defined by $|\xi| \leq \frac{1}{16}, |z| \leq \frac{1}{4}$.

## 4 Conclusion

We have added a result to the collection of the constructions of Lyapunov functions available in the literature. We have established a link between the center manifold theory and the Lyapunov approach. Time-varying and discrete-time versions of Theorem 1 can be proved.

## References

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