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Uniform exponential stability of linear time-varying systems: revisited

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Abstract

Some classical results known in the adaptive control literature are often used as analysis tools for *nonlinear* systems by evaluating the nonlinear differential equations along trajectories. While this technique is widely used, as we remark through examples, one must take special care in the consideration of the initial conditions in order to conclude uniform convergence. One way of taking care explicitly of the initial conditions is to study *parameterized* linear time-varying systems. This paper re-establishes known results for linear time-varying systems via new techniques while stressing the importance of imposing that the formulated sufficient and necessary conditions must hold uniformly in the parameter. Our proofs are based on modern tools which can be interpreted as an "integral" version of Lyapunov theorems; rather than on the concept of uniform complete observability which is most common in the literature. © 2002 Elsevier Science B.V. All rights reserved.

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Notations. In this note, $\|\cdot\|$ stands for the Euclidean norm of vectors and induced norm of matrices. $\|\cdot\|_\infty$ denotes the $\mathscr{L}_\infty$ norm of signals and $\|\cdot\|_p$, where $p\in[1,\infty]$, denotes the $\mathscr{L}_p$ norm of time signals. In particular, for a measurable function $\phi:\mathbb{R}_{\geqslant t_0}\to\mathbb{R}^n$, by $\|\phi\|_p$ we mean $(\int_{t_0}^{\infty}\|\phi(t)\|^p\,\mathrm{d}t)^{1/p}$ for $p\in[1,\infty)$ and $\|\phi\|_\infty$ denotes the quantity ess $\sup_{t\geqslant t_0}\|\phi(t)\|$. Unless otherwise specified we use, in general, the letter c to denote a positive constant. For positive definite matrices we use the bounds $p_m I\leqslant P\leqslant p_M I$. The solution of $\dot{x}=f(t,x)$ with initial conditions $(t_0,x_0)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}^n$, $x_0=x(t_0)$, is denoted $x(\cdot;t_0,x_0)$ or simply, $x(\cdot)$.

1. Introduction and motivations

Stability analysis and control design for families of systems (linear or nonlinear) has been extensively studied; specifically in the context of robust control. In the terms of [6], "a feedback law robustly [asymptotically] stabilizes a parameterized family of systems, if it [asymptotically] stabilizes *all* the systems in the family". Driven by problems stemming from adaptive control of linear systems and stabilization of *nonlinear* systems, in this paper we study

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the question of whether parameterized linear time varying systems (LTV) is uniformly exponentially stable where "uniform" is to be understood with respect to that parameter and the initial conditions. More specifically the common denominator in the two mentioned areas which motivates our study is that an ad hoc approach to stability analysis often undertaken in the literature, is to consider (under adequate boundedness assumptions nonlinear systems as linear time-varying, by evaluating the state variable along the trajectories (see for instance [10, pp. 626–627]). Such technique has been used for instance in the context of:

- *Robust output feedback control* (for feedback linearizable systems in [11,9]): In both references, the design approach consists of two basic steps: in the first, the authors define an adaptive state feedback controller which guarantees exponential convergence of the tracking and estimation errors; in the second, the output feedback controller is designed and, based on a converse Lyapunov function invoked for the state-feedback error dynamics, exponential convergence for the tracking and estimation errors is proved. However, we notice that this Lyapunov function is useful only if the exponential convergence is uniform.
- *Nonholonomic systems*: Samson proposed in [24], a change of coordinates which transforms a chained-form non-holonomic system (see [12]) into a skew-symmetric system $\dot{x} = P(t,x)x$, where $P(\cdot,\cdot) = -P(\cdot,\cdot)^{\mathrm{T}}$. In [17] we studied this form for 3-degrees of freedom systems by analyzing the system $\dot{\xi} = P(t, \xi(t,t_0,\xi_0))\xi$ which is parameterized by (t_0,ξ_0). Also, in [16] using the results as stated in this article we have been able to provide precise proofs of uniform convergence for systems of more than 3 states.
- *Model reference adaptive control* (MRAC) for linear time invariant systems (LTI): this consists on designing a controller so that the closed loop transfer function matches a "reference model". The central point to which we call the attention of the reader is that this technique typically yields a *nonlinear* time-varying closed loop system. To focus our discussion let us consider the following closed loop system encountered in MRAC (see e.g. [10, p. 635]):

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A & B\phi(t,z)^{\mathrm{T}} \\ -\phi(t,z)C^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix},$$

$$z := \begin{bmatrix} e \\ \theta \end{bmatrix}, \tag{1}$$

where $e \in \mathbb{R}^n$ represents a tracking error, $\theta \in \mathbb{R}^m$ a parameter estimation error and $\phi : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^{n+m} \to \mathbb{R}^m$ is a bounded smooth function, and the triple (A,B,C) is strictly positive real, i.e., satisfies the Kalman–Yakubovich–Popov (KYP) lemma (see [10, p. 407]). As it is well summarized for example in [25,1,8], if the regressor ϕ depends only on time, system (1) is uniformly (globally) exponentially stable if $\phi(t)$ is bounded, absolutely continuous and *persistently exciting* (PE), i.e., if there exist positive constants μ and T such that

$$\mu I \leqslant \int_t^{t+T} \phi(\tau)\phi(\tau)^{\mathrm{T}}\,\mathrm{d}\tau, \quad \forall t \geqslant 0. \tag{2}$$

However, when the function ϕ depends also on the state, this result is not applicable directly. As mentioned above, a solution often taken in the literature (see for instance [20,4,8,10,27,9]), is to construct an LTV system for each trajectory of (1). Then, under the assumption that the triple (A,B,C) is strictly positive real, exponential convergence of the state, is guaranteed by assuming that each trajectory $z(\cdot,t_0,z_0)$ produces a function

$$\tilde{\phi}(t) := \phi(t+t_0, z(t+t_0,t_0,z_0)) \tag{3}$$

satisfying (2). In this case, by defining the parameter $\lambda := (t_0,z_0)$ one can associate a parameterized LTV system to *each* pair of i.c. of the system (1):

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A & B\tilde{\phi}(t,\lambda)^{\mathrm{T}} \\ -\tilde{\phi}(t,\lambda)C^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix}, \quad z := \begin{bmatrix} e \\ \theta \end{bmatrix} \tag{4}$$

which has initial conditions $(t_\star,z_\star)$. Notice that in general, $(t_0,z_0) \neq (t_\star,z_\star)$ however, wherever the initial conditions of both systems (1) and (4) coincide, so do their solutions. The important implication of this is that, if *for each* λ the LTV system (4) is exponentially convergent, each trajectory of (1) converges exponentially fast to zero. If moreover, (4) is exponentially stable, uniformly in the parameter λ and in $(t_\star,z_\star)$

then (1) is also exponentially stable, uniformly in the initial conditions.

The difference between exponential convergence and uniform exponential stability is crucial. The former means that *for each* pair of initial conditions $(t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times B_r$ there exist γ_1 and $\gamma_2 > 0$, such that the solution $x(t, t_0, x_0)$ of the system satisfies

$$\|x(t, t_0, x_0)\| \leqslant \gamma_1 \|x_0\| \mathrm{e}^{-\gamma_2 (t - t_0)}. \tag{5}$$

In contrast to this, a system is called uniformly exponentially stable if and only if the constants in the bound above are independent of the initial conditions. In terms of robustness, examples in [19] show the following fact:

Fact 1. There exist systems $\dot{x} = f(t, x)$ such that $f(\cdot, \cdot)$ is locally Lipschitz, $f(\cdot, t)$ is globally Lipschitz uniformly in t, the origin is uniformly globally stable (UGS), the origin of the system is globally exponentially attractive but the system is *not* totally stable, that is, there exists bounded disturbances $d(\cdot)$ such that $\dot{x} = f(t, x) + d(t)$ becomes unstable.

1.1. Contributions of this paper

The discussion above motivates us to study the problem of uniform exponential stability of families of LTV systems.

While some of the results presented in this paper are well known in their version for non-parameterized systems, our main contribution relies in providing new proofs which involve modern tools that can be interpreted as an "integral" version of Lyapunov theorems and which do not rely on classical arguments such as uniform complete observability. As a byproduct, we are able to give explicit bounds for the convergence rates. The latter may be of interest specifically in identification and adaptive control.

Other proofs are based on performing first a Lyapunov transformation (which in our case of linear time-varying systems corresponds to a time-varying invertible mapping) in order to transform the system into a form which exhibits an exponentially stable "zero dynamics" and a "perturbing term" which depends on a bounded square-integrable output signal. We call this, "output injection", inspired by a similar technique used in the adaptive control textbooks, used to prove invariance of uniform complete observability. In this paper we prove (see Lemma 4) invariance of uniform exponential stability.

Our main conditions are stated in the form of persistency of excitation (PE), which is a well-known concept in the literature of adaptive and identification linear systems. However, we use a new definition of this property, for parameterized systems. As it will become clear from the sequel, our proofs can be carried out verbatim by ignoring the parameter λ, however, we find it worth re-stating certain known results and carrying out the proofs with this parameter in consideration since as we have pointed out before, the technique of analyzing a nonlinear system along trajectories is widely used but we have not been able to locate in the literature clear statements which take explicit care of this uniformity, for parameterized systems.

In particular, we give sufficient conditions for UGES of families of "skew-symmetric" systems, $\dot{x} = P(t, \lambda)x$, where $P(\cdot, \cdot) \in \mathbb{R}^{n \times n}$ is skew-symmetric, except for the element $[1, 1]$ which is non-zero (see Eq. (45)). The latter is an extension of the problem studied in [18] for positive semidefinite matrices $P(t)$. To the best of our knowledge, these results which are presented in Section 3.2, are new.

See also [17] which deals with uniform convergence of nonlinear systems under uniform persistency of excitation. Interestingly enough, the contents of this article are at the origins of the proofs of some of the results contained in [17]. See also [23].

The rest of the paper is organized in 3 sections. Next, we present some definitions and basic tools. The lemmas on output injection and integral conditions are presented in Section 3 followed by the restatement of classical results and two new results for stable systems. We conclude with some remarks in Section 4.

2. Preliminaries: definitions and tools

We start by stating the precise property we prove in this paper. [1] Let $\mathscr{D} \in \mathbb{R}^q$ be a closed not necessarily compact subset and $f : \mathbb{R}_{\geqslant 0} \times \mathscr{D} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, with $f(t, \lambda, \cdot)$ locally Lipschitz uniformly in t and λ. We will consider parameterized systems of

[1] Other definitions of stability for families of systems have been proposed e.g. in [15,3].

the form $\dot{x} = f(t, \lambda, x)$, i.e. systems characterized by a *constant* parameter λ.

Definition 1 (λ-UGES and ULES). The origin of the system $\dot{x} = f(t, \lambda, x)$ is said to be λ-ULES if there exist $r > 0$ and two constants k_λ and $\gamma_\lambda > 0$ such that, for all $t \geqslant t_0$, $\lambda \in \mathscr{D}$,

$$\|x_0\| < r \quad \Rightarrow \quad \|x(t, \lambda, t_0, x_0)\| \leqslant k_\lambda \|x_0\| \mathrm{e}^{-\gamma_\lambda (t - t_0)}.$$

Furthermore, the system is said to be λ-UGES if the exponential bound holds for all $(t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$.

From the definition above and the discussion in the Introduction it shall be clear that, for example the MRAC *nonlinear* system (1) is uniformly exponentially stable only if the associated system (4) is λ-UGES. As we will see, a system is λ-UGES only if the conditions imposed do not depend on the parameter λ.

In the same lines of this remark we present below a converse Lyapunov lemma which can be proven along the same lines as the proof of [10, Theorem 3.12] but clearly states that the converse Lyapunov function obtained is parameterized by λ but has uniform bounds. We state this result for further development.

Lemma 1 (Converse). *Let $\lambda \in \mathscr{D} \subset \mathbb{R}^p$ and let $\mathscr{A} : \mathbb{R}_{\geqslant 0} \times \mathscr{D} \to \mathbb{R}^{q \times q}$ be a continuous and globally bounded function. If the system $\dot{x} = \mathscr{A}(t, \lambda) x$ is* λ-UGES *then for any matrix function $Q(\cdot, \cdot)$ satisfying*

$$0 < c_3 I \leqslant Q(t, \lambda) \leqslant c_4 I \quad \forall (t, \lambda) \in \mathbb{R}_{\geqslant 0} \times \mathscr{D}, \tag{6}$$

there exist c_1 and $c_2 > 0$ such that the unique solution, $P_0(t, \lambda)$, of

$$\dot{P}_0(t, \lambda) + \mathscr{A}(t, \lambda)^{\mathrm{T}} P_0(t, \lambda) + P_0(t, \lambda) \mathscr{A}(t, \lambda) = -Q(t, \lambda),$$

$$P_0(t_0, \lambda) = I \tag{7}$$

satisfies

$$0 < c_1 I \leqslant P_0(t, \lambda) \leqslant c_2 I \quad \forall (t, \lambda) \in \mathbb{R}_{\geqslant 0} \times \mathscr{D}. \tag{8}$$

In particular,

$$V_0(t, \lambda, x) := x^{\mathrm{T}} P_0(t, \lambda) x \tag{9}$$

qualifies as a Lyapunov function for $\dot{x} = \mathscr{A}(t, \lambda) x$.

As mentioned in the Introduction, the main concept which links the results presented in this paper is persistency of excitation, a notion well established since more than 35 years. However, to be used for regressors possibly depending on *systems trajectories* we will introduce the following modified definition.

Definition 2 (λ-uniform persistency of excitation). Let the function $\phi : \mathbb{R}_{\geqslant 0} \times \mathscr{D} \to \mathbb{R}^{n \times m}$, be continuous. We say that $\phi(\cdot, \cdot)$ is λ-uniformly persistently exciting (λ-uPE) if there exist two parameters μ and $T > 0$ such that for each $\lambda \in \mathscr{D}$

$$\int_t^{t+T} \phi(\tau, \lambda) \phi(\tau, \lambda)^{\mathrm{T}} \, \mathrm{d}\tau \geqslant \mu I \quad \forall t \geqslant 0. \tag{10}$$

Clearly, the property above is more restrictive than the classical (non-uniform) PE but as we will see later, λ-uPE is necessary for uniform convergence. An important auxiliary tool which will allow us to establish shorter than the classical proofs (i.e., those based on uniform observability), is the following.

Lemma 2 (Measure). *Consider a function $\phi : \mathbb{R}_{\geqslant 0} \times \mathscr{D} \to \mathbb{R}$. Assume that there exists ϕ_M such that $\|\phi(t, \lambda)\| \leqslant \phi_M$ for all $t \geqslant 0$ and all $\lambda \in \mathscr{D}$. Assume further that $\phi(\cdot, \cdot)$ is* λ-uPE. *Then, for any $t \geqslant 0$ the measure of the set*

$$I_{\mu,t} := \left\{ \tau \in [t, t+T] \colon |\phi(\tau, \lambda)| \geqslant \frac{\mu}{2T\phi_M} \right\} \tag{11}$$

satisfies

$$\mathrm{meas}[I_{\mu,t}] \geqslant \sigma_\mu := \frac{T\mu}{2T\phi_M^2 - \mu}. \tag{12}$$

Proof. By assumption we have that

$$\mu \leqslant \int_t^{t+T} |\phi(\tau, \lambda)|^2 \, \mathrm{d}\tau \leqslant \int_{I_{\mu,t}} |\phi(\tau, \lambda)|^2 \, \mathrm{d}\tau + \int_{[t,t+T] \setminus I_{\mu,t}} \phi_M |\phi(\tau, \lambda)| \, \mathrm{d}\tau$$

and therefore

$$\mu - \int_{[t,t+T] \setminus I_{\mu,t}} \phi_M |\phi(\tau, \lambda)| \mathrm{d}\tau \leqslant \phi_M^2 \mathrm{meas}\,[I_{\mu,t}].$$

Then, from (11) we have that $|\phi(\tau, \lambda)| < \mu/(2T\phi_M)$ for all $\tau \in [t, t+T] \setminus I_{\mu,t}$ so

$$\mu - \frac{\mu}{2T}(T - \mathrm{meas}[I_{\mu,t}]) < \phi_M^2 \mathrm{meas}\,[I_{\mu,t}]$$

from which (12) follows. □

3. UGES for parameterized systems

We present in this section our main results. We will present new proofs for known theorems that establish λ-UGES for "strictly positive real" systems and the classical gradient adaptive algorithm. Also, we provide new results for "skew-symmetric" systems. In all cases, the system is known to be marginally stable and we will use the property of λ-uPE to show uniform exponential convergence. Rather than to the statements as such, we wish to call the attention of the reader towards the methods of proof. These rely on the Measure Lemma 2 and the following two simple but fundamental results.

In words, the first establishes λ-UGES under integral conditions. Then, Lemma 4 establishes that λ-UGES is invariant under output injection, provided *the output* is integrable. It is inspired by the classical result that, for linear time-varying systems, uniform complete observability is invariant under output injection (see e.g. [8, Lemma 4.8.1]). General versions of these lemmas have been presented in [26,22] for nonlinear (non-parameterized) systems. However, the proofs presented here are original and perhaps more significantly, for linear systems we are able to give precise convergence bounds.

Lemma 3 (Integral Lyapunov stability). *Assume that there exist constants $r, c, p > 0$ such that the solution*[2] $x(\cdot;\lambda,t_0,x_0)$ *of* $\dot{x} = f(t,\lambda,x)$ *satisfy*

$$\max\{\|x\|_\infty, \|x\|_p\} \leqslant c\|x_0\| \tag{13}$$

for all $x_0 \in B_r$ and all $t_0 \geqslant 0$. Then, the system is λ-ULES with $k_\lambda := c\mathrm{e}^{1/p}$ and $\gamma_\lambda := [pc^p]^{-1}$. Moreover, if $c > 0$ exists for all $x_0 \in R^n$, the system λ-UGES.

Proof. Firstly, the inequalities in (13) imply that for all $t \geqslant t_0 \geqslant 0$

$$\sup_{t\leqslant\tau}\|x(\tau)\|^p \leqslant c^p\|x(t)\|^p, \tag{14}$$

$$\int_t^\infty \|x(\tau)\|^p\,\mathrm{d}\tau \leqslant c^p\|x(t)\|^p. \tag{15}$$

Define $v(t) := \int_t^\infty \|x(\tau)\|^p\,\mathrm{d}\tau$ then $\dot{v} \leqslant -\frac{1}{c^p}v(t)$ which in turn implies by the comparison theorem that

$$v(t) \leqslant \exp\left(-\frac{(t-t_0)}{c^p}\right)v(t_0) \quad \forall t \geqslant t_0 \geqslant 0. \tag{16}$$

On the other hand, for any $T > 0$ we have that

$$\begin{aligned} T\|x(t+T)\|^p &\leqslant T\sup_{\tau\geqslant t+T}\|x(\tau)\|^p \\ &\leqslant \int_t^{t+T}\sup_{\tau\geqslant s}\|x(\tau)\|^p\,\mathrm{d}s \\ &\leqslant c^p\int_t^{t+T}\|x(s)\|^p\,\mathrm{d}s \end{aligned}$$

and by the definition of $v(\cdot)$,

$$T\|x(t+T)\|^p \leqslant c^p v(t),$$

so defining $T := c^p$, using (16) and $v(t_0) \leqslant c^p\|x_0\|^p$ we obtain that

$$\|x(t+T)\| \leqslant c\exp\left(-\frac{(t-t_0)}{pc^p}\right)\|x_0\| \quad \forall t \geqslant t_0 \geqslant 0. \tag{17}$$

The result follows observing that (13) implies that $\|x(t)\| \leqslant c\mathrm{e}^{1/p}\mathrm{e}^{-\gamma_\lambda(t-t_0)}\|x_0\|$ $\forall t \in [t_0, t_0+T]$ and (17) is the same as $\|x(t)\| \leqslant c\mathrm{e}^{-\gamma_\lambda(t-t_0-T)}$ for all $t \geqslant t_0+T$. □

Lemma 4 (Output injection). *Let $A : \mathbb{R}_{\geqslant 0} \times \mathscr{D} \to \mathbb{R}^{n\times n}$, $C : \mathbb{R}_{\geqslant 0} \times \mathscr{D} \to \mathbb{R}^{m\times n}$, $K : \mathbb{R}_{\geqslant 0} \times \mathscr{D} \to \mathbb{R}^{n\times m}$ be continuous and let $A(t,\lambda)$, $C(t,\lambda)$, $K(t,\lambda)$ be bounded for all $\lambda \in \mathscr{D}$ and all $t \geqslant 0$ by a bound independent of λ. Assume further that the system $\dot{\bar{x}} = A(t,\lambda)\bar{x}$ is λ-UGES. Then, the system $\dot{x} = A(t,\lambda)x + K(t,\lambda)y$ where $y := C(t,\lambda)x$, is λ-UGES if there exists $c > 0$ such that*

$$\|y\|_2 \leqslant c\|x_0\| \quad \forall (t_0,x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n. \tag{18}$$

Proof. From Lemma 1 there exists $V_0(t,\lambda,x)$ satisfying (6)–(9), hence

$$\dot{V}_0(t,\lambda,x(t)) \leqslant -c_3\|x(t)\|^2 + c_2k_M\|x(t)\|\|y(t)\|, \tag{19}$$

where we used $\|P_0(t,\lambda)\| \leqslant c_2$ and $\|K(t,\lambda)\| \leqslant k_M$ for all t and all λ. We now show that the system is uniformly stable. Using the inequality $ab \leqslant 0.5\varepsilon^2\|a\|^2 + 1/(2\varepsilon^2)\|b^2\|$ with $\varepsilon = [2(1-\rho)c_3]^{1/2}$, $\rho < 1$ to bound

[2] For simplicity we assume here that $f(\cdot,\lambda,\cdot)$ is such that solutions exist and are unique. However, the same result can be proven for the case of multiple solutions, see [26,22].

the last term in (19) we obtain

$$\dot{V}_0(t,\lambda,x(t)) \leqslant -\rho c_3\|x(t)\|^2 + \frac{[c_2k_M]^2}{4(1-\rho)c_3}\|y(t)\|^2. \tag{20}$$

Then, integrating on both sides and using (8) and (18) we obtain that

$$\|x\|_\infty \leqslant \left(\frac{\pi}{c_1}\right)^{1/2}\|x_0\|, \quad \pi := c_2 + c^2\,\frac{(c_2k_M)^2}{4(1-\rho)c_3}. \tag{21}$$

Coming back to (20) and using (18) we also have that for all $t \geqslant t_0$,

$$\rho c_3\|x\|_2^2 \leqslant V_0(t_0,\lambda,x(t_0)) + c^2\,\frac{[c_2k_M]^2}{4(1-\rho)c_3}\|y\|_2^2$$

so we conclude from (8) that

$$\|x\|_2 \leqslant \left(\frac{\pi}{\rho c_3}\right)^{1/2}\|x_0\|.$$

The proof finishes invoking Lemma 3 with $p=2$. □

Remark 1. Notice that Lemma 3 also yields explicit convergence bounds. In particular, for sufficiently small $\rho \leqslant c_1/c_3$ we obtain that

$$k_\lambda := \left(\frac{\pi \mathrm{e}}{\rho c_3}\right)^{1/2}, \quad \gamma_\lambda := \left(\frac{2\pi}{\rho c_3}\right)^{-1}. \tag{22}$$

We will use this lemma to show the utility of the *output injection technique* in proving two theorems for marginally stable parameterized systems. The approach consists on performing a Lyapunov transformation to put the system in the output injection form of Lemma 4. Such technique has been used in other papers dealing with nonlinear systems, such as [14,2]. The main difference with respect to those works is that here we aim at verifying the integral conditions of Lemma 3 below instead of finding a suitable Lyapunov function.

3.1. Some "classical" theorems revisited

We will restate and reprove two results well known in the adaptive control literature: "SPR" systems and the "gradient algorithm" (see e.g. [20,25,8,1]). A novelty of our presentation is that the use of Lemmas 3 and 4 yield constructive proofs, in particular we are able to give explicit convergence bounds, as opposed to a proof involving uniform observability and output injection. Moreover, our proofs are considerably more compact than those which rely on showing the equivalence of PE and uniform observability and the invariance of the latter under output injection. For the sake of comparison see for instance [8, Proof of Theorem 4.3.2(iii), p. 236] which requires [8, Theorem 3.4.8, Lemma 4.8.1]. See also [1, Section 2.3] and [25, pp. 73–74, 332–335]. In addition, see the recent [3] result [5] which gives convergence rates for the gradient algorithm.

Lemma 5 (Gradient algorithm). *For the system*

$$\dot{x} = -\Phi(t,\lambda)\Phi(t,\lambda)^{\mathrm{T}}x \tag{23}$$

where $\lambda \in \mathscr{D}$, *assume that* $\Phi(\cdot,\cdot)$ *is* λ-*uPE with parameters* T *and* $\mu > 0$ *and there exists a constant* $\phi_M > 0$ *such that for all* $t \geqslant 0$ *and all* $\lambda \in \mathscr{D}$, $\|\Phi(t,\lambda)\| \leqslant \phi_M$. *Then* (23) *is* λ-*UGES with* $k_\lambda = 1$, $\gamma_\lambda \geqslant \mu/T(1+\phi_M^2T)^2$.

Remark 2. It is clear from the claim above that if the function $\Phi(\cdot,\cdot)$ is not λ-uPE, the convergence rate depends on the parameter λ. This observation is particularly important when considering the system (23) to conclude stability and convergence for the *nonlinear* system $\dot{\bar{x}} = -\bar{\Phi}(t,\bar{x})\bar{\Phi}(t,\bar{x})^{\mathrm{T}}\bar{x}$ by defining $\Phi(t,\lambda) := \bar{\Phi}(t,\bar{x}(t,\lambda))$ and $\lambda := (t_0,\bar{x}_0)$. In view of this and the discussion of the Introduction, it may be clear that the origin $\bar{x}=0$ is exponentially attractive, uniformly in the initial conditions, if and only if the function $\bar{\Phi}(t,\bar{x}(t,\lambda))$ is persistently exciting, uniformly in the initial conditions $(t_0,\bar{x}_0)$, i.e. if it is λ-uPE. The same claim is valid for all the other results presented in this paper.

Sketch of proof. Let $V(t,x) := \frac{1}{2}\|x\|^2$, then

$$\dot{V}(\tau,x(\tau,\lambda)) = -x(\tau,\lambda)^{\mathrm{T}}\Phi(\tau,\lambda)\Phi(\tau,\lambda)^{\mathrm{T}}x(\tau,\lambda) \tag{24}$$

therefore, defining $v(t,\lambda) := V(t,x(t,\lambda))$, we have that $v(t,\lambda) \leqslant \frac{1}{2}\|x_0\|^2$ and

$$v(t+T,\lambda) - v(t,\lambda) = -\int_t^{t+T}\|\Phi(\tau,\lambda)^{\mathrm{T}}x(\tau,\lambda)\|^2\,\mathrm{d}\tau, \tag{25}$$

[3] The cited paper was published after the original submission of this work.

where the solution

$$x(\tau,\lambda) = x(t,\lambda) - \int_t^\tau \Phi(s,\lambda)\Phi(s,\lambda)^{\mathrm{T}} x(s,\lambda)\,\mathrm{d}s. \quad (26)$$

Substituting (26) in (25), using $(a-b)^2 \geqslant [\rho/(1+\rho)]/a^2 - \rho b^2$, the triangle inequality, Cauchy–Schwartz inequality and the λ-uPE property successively we obtain that

$$\frac{\mu\rho}{1+\rho}\|x(t,\lambda)\|^2 \leqslant (1+\rho\phi_M^4 T^2)[v(t,\lambda) - v(t+T,\lambda)] \quad (27)$$

which, together with $0.5\|x(t,\lambda)\|^2 = v(t,\lambda) \leqslant 0.5\|x_0\|^2$, implies that

$$v(t+T,\lambda) \leqslant (1-\sigma)v(t,\lambda),$$

$$\sigma := \frac{2\rho\mu}{(1+\rho)(1+\rho\phi_M^4 T^2)}, \quad (28)$$

where obviously, ρ is taken so that $1 > \sigma > 0$. Then, defining

$$\gamma_\lambda := -\frac{\ln(1-\sigma)}{T} \geqslant \frac{\sigma}{T} \quad \text{and} \quad \rho := \frac{1}{\phi_M^2 T}$$

it is straightforward to show that

$$v(t,\lambda) \leqslant v(t_0,\lambda)\mathrm{e}^{-\gamma_\lambda(t-t_0)}$$

and the result follows.

Remark 3.

- The sketch of proof above follows along similar lines as the proof of [20, Theorem 2.16]. For a complete proof see [21].
- Notice that different explicit bounds can be obtained integrating (27) and invoking Lemma 3.
- Finally, it is interesting to see that one more alternative proof of Lemma 5 can be established using Lemma 4 observing that the λ-uPE assumption is equivalent to uniform complete [4] observability of the system

$$\dot{x} = 0 \quad (29)$$

$$y = \Phi(t,\lambda)^{\mathrm{T}} x \quad (30)$$

so it follows [5] that there exists a continuous bounded function $K(\cdot,\cdot)$ such that the system

$$\dot{x} = K(t,\lambda)\Phi(t,\lambda)^{\mathrm{T}} x$$

is λ-UGES. The result follows invoking Lemma 4 on the system

$$\dot{x} = K(t,\lambda)\Phi(t,\lambda)^{\mathrm{T}} x + [K(t,\lambda) - \Phi(t,\lambda)]y,$$

$$y = \Phi(t,\lambda)^{\mathrm{T}} x.$$

While this seems to be a very direct proof, in general $K(\cdot,\cdot)$ above it is not known and therefore one cannot establish a bound on the rate of convergence which is often desirable in adaptive control applications.

The second, and more interesting theorem that we revisit applies to a class of systems which includes (4) and arises in adaptive control theory. We offer a short proof of sufficiency for uniform exponential stability under uniform persistency of excitation. We consider parameterized LTV multivariable systems of the form

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A(t,\lambda) & B(t,\lambda)^{\mathrm{T}} \\ -C(t,\lambda) & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix}, \quad z := \begin{bmatrix} e \\ \theta \end{bmatrix}, \quad (31)$$

where $e \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$, $A(t,\lambda) \in \mathbb{R}^{n\times n}$, $B(t,\lambda) \in \mathbb{R}^{m\times n}$, $C(t,\lambda) \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathscr{D} \subset R^l$.

Theorem 1 (UGES of LTV). *Consider the system* (31) *under the following assumptions*:

Assumption 1. *There exists $\phi_M > 0$ such that for all $t \geqslant 0$ and for all $\lambda \in \mathscr{D}$,*

$$\max\left\{\|B(t,\lambda)\|, \left\|\frac{\partial B(t,\lambda)}{\partial t}\right\|\right\} \leqslant \phi_M. \quad (32)$$

Assumption 2. *There exist symmetric matrices $P(t,\lambda)$ and $Q(t,\lambda)$ such that $P(t,\lambda)B(t,\lambda)^{\mathrm{T}} = C(t,\lambda)^{\mathrm{T}}$ and $-Q(t,\lambda) := A(t,\lambda)^{\mathrm{T}}P(t,\lambda) + P(t,\lambda)A(t,\lambda) + \dot{P}(t,\lambda)$. Furthermore, $\exists\, p_m, q_m, p_M$, and $q_M > 0$ such*

[4] Here, the qualifier “uniform” refers also to the parameter λ. We recall that for applications to nonlinear systems λ corresponds to the initial times and states of the nonlinear system in question.

[5] Based for instance on [7].

that, for all $(t,\lambda)\in\mathbb{R}_{\geqslant 0}\times\mathscr{D}$, $p_m I\leqslant P(t,\lambda)\leqslant p_M I$ *and* $q_m I\leqslant Q(t,\lambda)\leqslant q_M I$.

Then, the system is λ-UGES *if and only if* $B(\cdot,\cdot)$ *is* λ-uPE.

In words, we claim that one cannot obtain uniform exponential stability of (31) if μ and T in (10) depend on λ. As discussed before, even if the inclusion of a parameter might seem annecdotic in the linear context, this elementary observation cannot be overestimated when dealing with nonlinear systems in the way described in Remark 2 and the Introduction.

Proof of Theorem 1 (*Sufficiency*). The proof will follow by applying the Output Injection Lemma 4 and the coordinate transformation

$$\xi := \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = \begin{bmatrix} I & 0\\ -B(t,\lambda) & I\end{bmatrix} z. \tag{33}$$

It is clear that under Assumption 1 there exist constants t_M and $t_M^{-1}>0$ such that

$$\|\xi\|\leqslant t_M\|z\|,\qquad \|z\|\leqslant t_M^{-1}\|\xi\|\quad \forall (t,\lambda)\in\mathbb{R}_{\geqslant 0}\times\mathscr{D} \tag{34}$$

in view of which, the system (31) is λ-UGES if and only if the following system is λ-UGES:

$$\begin{bmatrix}\dot\xi_1\\ \dot\xi_2\end{bmatrix} = \underbrace{\begin{bmatrix} A(t,\lambda) & B(t,\lambda)^{\mathrm T}\\ -R_1(t,\lambda) & -B(t,\lambda)B(t,\lambda)^{\mathrm T}\end{bmatrix}}_{\mathscr{A}(t,\lambda)}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + \underbrace{\begin{bmatrix} B(t,\lambda)^{\mathrm T}B(t,\lambda)\\ R_1(t,\lambda)-R_2(t,\lambda)\end{bmatrix}}_{\mathscr{K}(t,\lambda)}\xi_1, \tag{35}$$

where $R_1(t,\lambda):=P_B^{-1}(t,\lambda)B(t,\lambda)P(t,\lambda)$ and $R_2(t,\lambda):=\dot B(t,\lambda)+B(t,\lambda)[P(t,\lambda)+A(t,\lambda)+B(t,\lambda)^{\mathrm T}B(t,\lambda)]$ and P_B is a positive definite symmetric matrix to be defined.

To prove that (35) is λ-UGES we will use Lemma 4 with output ξ_1. To that end, assume for the time being that $\dot\zeta =: \mathscr{A}(t,\lambda)\zeta$ is λ-UGES. Next, consider the Lyapunov function $V_1(t,\lambda,z):=\frac{1}{2}z^{\mathrm T}R(t,\lambda)z$, with $R:=$ block-diag$\{P(t,\lambda),I\}$. Its total time derivative along (31) yields $\dot V_1(t,\lambda,z)\leqslant -q_m\|e\|^2$. Hence $\|e\|_2\leqslant c_*\|z_0\|$ with $c_*:=\sqrt{(p_M+1)/q_m}$. Then, using (34) we also have that $\|\xi_1\|_2\leqslant c_* t_M^{-1}\|\xi_0\|$. Moreover, under Assumptions 2 and 1 there exists k_M such that $\|\mathscr{K}(t,\lambda)\|\leqslant k_M$ for all $\lambda\in\mathscr{D}$ and all $t\geqslant 0$. Hence, invoking Lemma 4 we obtain that

$$\|\xi(t)\|\leqslant\left(\frac{\pi e}{\rho c_3}\right)^{1/2}\exp\left[-\left(\frac{2\pi}{\rho c_3}\right)^{-1}(t-t_0)\right], \tag{36}$$

where π is given in (21) with $c:=c_* t_M^{-1}$ and c_2,c_3 are generated via Lemma 1 by the λ-UGES property of $\dot\zeta =: \mathscr{A}(t,\lambda)\zeta$. It is only left to prove that $\dot\zeta =: \mathscr{A}(t,\lambda)\zeta$ is λ-UGES.

For this, we will argue as follows: from Lemma 5, the system $\dot{\bar\theta}=-B(t,\lambda)B(t,\lambda)^{\mathrm T}\bar\theta$ is λ-UGES therefore, from Lemma 1 there exists $P_B(t,\lambda)=P_B(t,\lambda)^{\mathrm T}>0$ for all $(t,\lambda)\in\mathbb{R}_{\geqslant 0}\times\mathscr{D}$ such that

$$\dot P_B(t,\lambda)-P_B(t,\lambda)B(t,\lambda)B(t,\lambda)^{\mathrm T} -B(t,\lambda)B(t,\lambda)^{\mathrm T}P_B(t,\lambda)=-I\quad\forall (t,\lambda)\in\mathbb{R}_{\geqslant 0}\times\mathscr{D}. \tag{37}$$

The result follows using the Lyapunov function $V(t,\lambda,\zeta):=\zeta_1^{\mathrm T}P(t,\lambda)\zeta_1+\zeta_2^{\mathrm T}P_B(t,\lambda)\zeta_2$ whose time derivative along $\dot\zeta =: \mathscr{A}(t,\lambda)\zeta$, using Assumption 2 and (37), yields $\dot V(t,\lambda,\zeta)=-\zeta_1^{\mathrm T}Q(t,\lambda)\zeta_1-\zeta_2^{\mathrm T}\zeta_2$.

Necessity: This part follows for instance as in [18]. We refer the reader to [21]. □

Remark 4. It is worth remarking that explicit bounds on P_B and Q can be obtained as in the proof of [10, Theorem 3.12] based on the parameters μ and T generated by the λ-uPE property of $B(\cdot,\cdot)$. This exhibits through (36), the dependence of the convergence rate and overshoot, on μ and T.

3.2. Other applications

We now address the problem of proving uniform convergence for time-varying systems with a Lyapunov function whose derivative possess a bound with time-varying non-positive coefficients. In particular, we will show λ-UGES for "skew-symmetric" systems. Examples of nonlinear systems which lay in this class (when regarded as linear time-varying in the sense discussed in the Introduction) includes non-holonomic systems [17,24,13]. The novelty of our result is that we prove uniform exponential stability, by using an inductive procedure in which at each step, we perform a Lyapunov transformation that yields an UGS system of the "output injection" form of Lemma 4. $n-1$ such transformations are realized until one obtains a "skew-symmetric" system of the form (48), "perturbed" by an output injection term. We start with the following.

Lemma 6. *Consider a forward complete system $\dot{x} = f(t,\lambda,x)$ with unique solutions for each initial condition. Assume that there exist reals $p_i \geqslant 1$, $i \leqslant n$ and a smooth function $V : \mathbb{R}_{\geqslant 0} \times \mathscr{D} \times \mathbb{R}^n$ such that*[6]

$$c_1\|x\|^2 \leqslant V(t,\lambda,x) \leqslant c_2\|x\|^2, \tag{38}$$

$$\dot{V}(t,\lambda,x) \leqslant -\sum_{i=1}^{n} |\phi(t,\lambda)|^{p_i} x_i^2, \tag{39}$$

where $\phi(t,\lambda)$ is λ-uPE and bounded for all $(t,\lambda) \in \mathbb{R}_{\geqslant 0} \times \mathscr{D}$. Then, the system is λ-UGES with

$$k_\lambda := \max\left\{\frac{T(c_3+1)}{c_3}, \frac{c_2}{c_1}\right\}\sqrt{\mathrm{e}}$$

$$\gamma_\lambda := \frac{1}{2}\max\left\{\frac{T(c_3+1)}{c_3}, \frac{c_2}{c_1}\right\}^{-2}, \tag{40}$$

where c_3 depends on μ, T and ϕ_M and is defined below.

Proof. The proof follows by appealing to Lemma 2 to prove that $x(t)$ is square integrable and then invoking Lemma 3. To that end, we first stress that if the *scalar* function $\phi(t,\lambda)$ is λ-uPE then $\psi(t,\lambda) := \phi(t,\lambda)^p$ is λ-uPE for any $p > 1$ with parameters T and $\mu_p := \mu^p/T^{p/q}$ where $1/p + 1/q = 1$. This fact follows from Cauchy–Schwartz inequality noticing that for each $p > 1$ there exists q satisfying $1/p + 1/q = 1$ and

$$\mu \leqslant \int_t^{t+T} \phi(\tau,\lambda)^2\,\mathrm{d}\tau \leqslant \left(\int_t^{t+T} \phi(\tau,\lambda)^{2p}\,\mathrm{d}\tau\right)^{1/p} T^{1/q}.$$

Therefore, defining

$$\bar{\mu} := \min_{\substack{(p_i,q_i)\\ p_i>1}} \left\{\frac{\mu^{p_i}}{T^{p_i/q_i}}, \mu\right\} \tag{41}$$

using (39) and invoking Lemma 2 we have that $v(t,\lambda) := V(t,x(t,\lambda))$ satisfies

$$v(t+T,\lambda) - v(t,\lambda) \leqslant -\frac{\bar{\mu}}{2T\phi_M}\int_{I_{\bar{\mu},t}} \|x(\tau,\lambda)\|^2\,\mathrm{d}\tau.$$

Also, from (39) it follows directly that $\|x\|_\infty \leqslant (c_2/c_1)\|x_0\|$ and therefore $\tau \in I_{\bar{\mu},t}$ implies that $\tau \leqslant t+T$. Then we have that $\|x(t+T,\lambda)\| \leqslant (c_2/c_1)\|x(\tau,\lambda)\|$ for all $\tau \in I_{\bar{\mu},t}$, and

$$\begin{aligned} v(t+T,\lambda) - v(t,\lambda) &\leqslant -\frac{\bar{\mu}c_1}{2c_2T\phi_M}\int_{I_{\bar{\mu},t}} \|x(t+T,\lambda)\|^2\,\mathrm{d}\tau \\ &\leqslant -\frac{c_1\bar{\mu}\sigma_{\bar{\mu}}}{2c_2T\phi_M}\|x(t+T,\lambda)\|^2. \end{aligned} \tag{42}$$

Integrating from t_0 to ∞ on both sides we obtain that

$$\int_{t_0}^{t_0+T} v(t,\lambda)\,\mathrm{d}t \geqslant c_3\int_{t_0+T}^{\infty} \|x(s,\lambda)\|^2\,\mathrm{d}s, \tag{43}$$

where we defined $c_3 := c_1\bar{\mu}\sigma_{\bar{\mu}}/2c_2T\phi_M$. Observing that $v(t,\lambda) \leqslant c_2\|x_0\|^2$ for all $t \geqslant t_0$, we make use of (43) and the inequality $\|x\|_\infty \leqslant c_2/c_1\|x_0\|$ to obtain

$$\|x\|_2^2 < \frac{T(c_3+1)}{c_3}\|x_0\|^2.$$

The proof finishes invoking Lemma 3 with $p = 2$ and $c := \max\{T(c_3+1)/c_3, c_2/c_1\}$. □

Remark 5. We remark that the result in the lemma above can also be shown with other bounds and in a more direct manner however, we favour the proof given above to illustrate the utility of Lemma 2. Indeed, we only need to observe that (39) implies that

$$\dot{V}(t,\lambda,x) \leqslant -\frac{\phi_M^{\min\{p_i\}}}{c_2\phi_M^{\max\{p_i\}}}\|\phi(t,\lambda)\|^{\max\{p_i\}}V(t,\lambda,x)$$

and using Gronwall–Bellman's inequality we obtain that,

$$v(t+T,\lambda) \leqslant \mathrm{e}^{-c\int_t^{t+T}\|\phi(t,\lambda)\|^{\max\{p_i\}}}v(t,\lambda),$$

$$c := \frac{\phi_M^{\min\{p_i\}}}{c_2\phi_M^{\max\{p_i\}}}$$

hence,

$$v(t+T,\lambda) \leqslant \mathrm{e}^{-c\mu_{\max p_i}}v(t,\lambda)$$

[6] Obviously, we mean $\dot{V} := \partial V/\partial t + \partial V/\partial x f$.

which implies the exponential convergence of $v(t,\lambda)$ for any $c_2>0$.

An interesting application of the above is the following.

Theorem 2. *Let the function $\phi:\mathbb{R}_{\geqslant 0}\times\mathscr{D}\to\mathbb{R}$ be λ-uPE and $\phi(\cdot,\lambda)$ be once-differentiable.*[7] *Assume further that there exist $\phi_M>0$ such that*

$$\max\{\|\phi(t,\lambda)\|,\|\dot\phi(t,\lambda)|\}\leqslant\phi_M \tag{44}$$

for all $t\geqslant 0$ and for all $\lambda\in\mathscr{D}$. Then, the system

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \vdots\\ \dot x_n\end{bmatrix}=\begin{bmatrix}-\phi(t,\lambda)^2 & \phi(t,\lambda) & 0 & \cdots & 0\\ -\phi(t,\lambda) & 0 & \phi(t,\lambda) & 0 & \vdots\\ 0 & -\phi(t,\lambda) & 0 & \ddots & 0\\ \vdots & 0 & \ddots & \ddots & \phi(t,\lambda)\\ 0 & \cdots & 0 & -\phi(t,\lambda) & 0\end{bmatrix}\times\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_n\end{bmatrix} \tag{45}$$

is λ-UGES.

Remark 6. It shall be clear from the proof that one can replace the elements in the second diagonals in the matrix above, by $\phi_1(t,\lambda),\dots,\phi_{n-1}(t,\lambda)$ and their negative counterpart, where each function $\phi_i(\cdot,\lambda)$ is once-differentiable, λ-uPE and satisfies a bound like (44). Also, the result follows without much difficulty if one multiplies each of the elements on one of the second diagonals by different constant coefficients.

Proof of Theorem 2. The proof consists on applying recursively Lemma 4 and once Lemma 6. In the sequel, for notational simplicity we will drop the arguments (t,λ).

1. For system (45) define the output $y_1:=\phi x_1$.
2. Let $V:=\frac{1}{2}\|x\|^2$. Then $\dot V=-y_1^2$ and therefore $\|y_1\|_2\leqslant\|x_0\|$ and $\|x\|_\infty\leqslant\|x_0\|$.
3. Consider the coordinate transformation ($z:=T(\phi)x$)

$$z:=\begin{bmatrix}z_1\\ z_2\\ \vdots\\ \vdots\\ z_n\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ -\phi^3 & 1 & 0 & \ddots & \vdots\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & \cdots & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ \vdots\\ x_n\end{bmatrix}$$

then system (45) is equivalent to

$$\underbrace{\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot x_3\\ \vdots\\ \vdots\\ \dot x_n\end{bmatrix}}_{\dot z}=\underbrace{\begin{bmatrix}-\phi^2 & \phi & 0 & \cdots & \cdots & 0\\ -\phi & -\phi^4 & \phi & \ddots & \ddots & \vdots\\ 0 & -\phi & 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \phi\\ 0 & \cdots & \cdots & 0 & -\phi & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ x_3\\ \vdots\\ \vdots\\ x_n\end{bmatrix}}_{A_2(t,\lambda)z}+\underbrace{\begin{bmatrix}\phi^3\\ -3\phi\dot\phi+\phi^4-\phi^6\\ -\phi^3\\ 0\\ \vdots\\ 0\end{bmatrix}}_{K_2(t,\lambda)}y_1. \tag{46}$$

4. Notice that there exist two constants $c_{T1}(\phi_M)$ and $c_{T2}(\phi_M)>0$ s.t. $\|z\|\leqslant c_{T1}\|x\|$ and $\|x\|\leqslant c_{T2}\|z\|$, therefore system (46) is UGS, i.e., $\|z\|_\infty\leqslant c_{T1}c_{T2}\|z_0\|$.
5. Since the output $y_1^*:=\phi z_1$ of the system (46) coincides with the output y_1 of (45) i.e., $y_1\equiv y_1^*$ we have that $\|y_1^*\|_2\leqslant\|x_0\|$. From item 4 we conclude that $\|y_1\|_2^*\leqslant c_{T2}\|z_0\|$.
6. Observing that $K_2(t,\lambda)$ is uniformly bounded we can use Lemma 4 to obtain that the system (46) is λ-UGES if the system $\dot z=A_2(t,\lambda)z$ is λ-UGES.

In order to prove λ-UGES of $\dot x=A_2(t,\lambda)x$ we repeat the steps above with the output $y_2:=\mathrm{col}[\phi x_1,\phi^2x_2]$ which, with the same function V as above, satisfies $\|y_2\|_2\leqslant\|x_0\|$. We define the new change of coordinates $z_3=x_3-\phi^5 z_2$ and proceed as above to find that the system $\dot x=A_2(t,\lambda)x$ is equivalent to a system of the form $\dot z=A_3(t,\lambda)z+K_3(t,\lambda)y_2$, where

[7] Differentiability is assumed here for simplicity but one can relax this assumption to absolute continuity and bounded variation by making the pertinent modifications to the previous Lemmas.

$[A_3(t,\lambda)]_{(i,j)} = [A_2(t,\lambda)]_{(i,j)}$ for all $i,j \leqslant n$, $(i,j) \neq (3,3)$ and $[A_3(t,\lambda)]_{(3,3)} = -\phi^6$, $y_2 := \mathrm{col}[\phi z_1, \phi^2 z_2]$. Long but straightforward calculations show that $K_3(t,\lambda)$ depends only on $\phi(t,\lambda)$ and $\dot{\phi}(t,\lambda)$, hence it is uniformly bounded due to (44). The proof reduces to show that $\dot{z} = \mathscr{A}_3(t,\lambda)z$ is λ-UGES.

The same procedure is repeated for each system $\dot{x} = A_i(t,\lambda)x$ up to $i = n-1$. At this step we have that the system $\dot{x} = A_{n-1}(t,\lambda)x$ is equivalent to

$$\dot{z} = A_n(t,\lambda)z + \underbrace{\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \phi^{n+1} \\ 0 & \cdots & \phi^{n+2} & -(2n-1)\phi^{n-1}\dot{\phi} + \phi^{3n-2} - \phi^{3n-1} \end{bmatrix}}_{K_n(t,\lambda)} \times \begin{bmatrix} (y_{n-1})_1 \\ \vdots \\ (y_{n-1})_{n-1} \end{bmatrix}, \tag{47}$$

where and $y_{n-1} := \mathrm{col}[\phi z_1, \ldots, \phi^{n-1} z_{n-1}]$ and $\dot{z} = A_n(t,\lambda)z$ corresponds to

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} -\phi(t,\lambda)^2 & \phi(t,\lambda) & 0 & \cdots & 0 \\ -\phi(t,\lambda) & -\phi(t,\lambda)^4 & \phi(t,\lambda) & 0 & \vdots \\ 0 & -\phi(t,\lambda) & -\phi(t,\lambda)^6 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \phi(t,\lambda) \\ 0 & \cdots & 0 & -\phi(t,\lambda) & -\phi(t,\lambda)^{2n} \end{bmatrix} \times \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} \tag{48}$$

which is λ-UGES by virtue of Lemma 6 (with $V(t,\lambda,z) := 0.5\|z\|^2$). Therefore, the system (47) is λ-UGES if $\dot{z} = A_n(t,\lambda)z$ is λ-UGES. □

4. Conclusions

We have presented new proofs to classical results on uniform global exponential stability of multivariable linear time-varying systems. The results are stated in terms of a reformulation of the well-known concepts of persistency of excitation. To best of our knowledge the proofs, which are based on Lyapunov transformations and the observation that UGES is invariant under output injection, are novel.

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