Massive MIMO Radar for Target Detection

S. Fortunati, L. Sanguinetti, F. Gini, M. S. Greco

Dipartimento di Ingegneria Informazione, University of Pisa, IT

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Outline[1]

- **Motivation**: a robust target detection problem.
- **Main goal**: derive a detector whose asymptotic distribution is invariant with respect to the true, but generally unknown, probability density function (pdf) of the disturbance.
- **How to achieve it**: increase the spatial degrees of freedom (DoF) using a co-colocated MIMO radar.

[1] Additional details regarding the material of this presentation can be found in:
S. Fortunati, L. Sanguinetti, F. Gini, M. S. Greco, and M. Himed, "Massive MIMO radar for target detection", submitted to *IEEE Transactions on Signal Processing*, 2019 (Available at: https://arxiv.org/abs/1906.06191).

The target detection problem

- Consider a multiple antenna radar system with N spatial channels, collecting K temporal snapshots $\{\mathbf{x}_k\}_{k=1}^K \in \mathbb{C}^N$.

- **Detection problem**:

$$
\begin{array}{lll}
H_0: & \mathbf{x}_k = \mathbf{c}_k & k = 1, \ldots, K, \\
H_1: & \mathbf{x}_k = \alpha_k \mathbf{v}_k + \mathbf{c}_k & k = 1, \ldots, K,
\end{array}
$$

  - $\mathbf{v}_k \in \mathbb{C}^N$: known at each time instant $k \in \{1, \ldots, K\}$,
  - $\alpha_k \in \mathbb{C}$: deterministic, *unknown*, scalar that may vary over k,
  - $\mathbf{C} \triangleq [\mathbf{c}_1, \ldots, \mathbf{c}_k]$: disturbance.

- A decision statistic $\Lambda(\mathbf{X})$ needs to be implemented:

$$
\Lambda(\mathbf{X}) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda, \qquad \mathbf{X} \triangleq [\mathbf{x}_1, \ldots, \mathbf{x}_K].
$$

How to choose the threshold

- The threshold λ should be chosen to maintain the P_{FA} below a pre-assigned value:

$$\Pr\left\{\Lambda(\mathbf{X}) > \lambda | H_0\right\} = \int_{\lambda}^{\infty} p_{\Lambda|H_0}(a|H_0)da = \overline{P_{FA}}.$$

- $p_{\Lambda|H_0}$ is the pdf of $\Lambda(\mathbf{X})$ under the null hypothesis H_0.

- Three simplifying assumptions are generally adopted:
  - M1 $\{\mathbf{c}_k\}_{k=1}^{K}$ are i.i.d. over the observation interval,
  - M2 α_k maintains constant over k: $\alpha_k \equiv \alpha$, $\forall k$,
  - M3 The pdf $p_{\mathbf{C}}(\mathbf{C}) = \prod_{k=1}^{K} p_C(\mathbf{c}_k)$ is perfectly known.

Perfectly matched GLR

- Under M1, M2 and M3, the Generalized Likelihood Ratio (GLR) statistic $\Lambda_{\mathrm{GLR}}(\mathbf{X})$ can be derived.

- Under H_0, as the number of temporal snapshots grows to infinity ($K \to \infty$), we get:[2]

$$\Lambda_{\mathrm{GLR}}(\mathbf{X}|H_0) \underset{K\to\infty}{\sim} \chi_2^2(0).$$

- Consequently, an asymptotic solution for λ is: $\bar{\lambda} = -2\ln \overline{P_{FA}}$.

Is it possible to derive a detection statistic with the same asymptotic properties of $\Lambda_{\mathrm{GLR}}(\mathbf{X})$ without relying on Assumptions M1, M2 and M3?

[2] S. M. Kay, *Fundamentals of statistical signal processing, volume II: detection theory*, Prentice Hall, 1993.

Spatial asymptotic regime

- We collect a single temporal snapshot ($K = 1$) and exploit the spatial dimension N:

$$
\begin{aligned}
H_0 &: \quad \mathbf{x} = \mathbf{c} \\
H_1 &: \quad \mathbf{x} = \alpha \mathbf{v} + \mathbf{c},
\end{aligned}
$$

- This allows us to entirely drop Assumptions M1 and M2.

Note that, unlike in the temporal domain, the spatial samples $x_1, \ldots, x_N$ *cannot* be considered as *independent* observations!

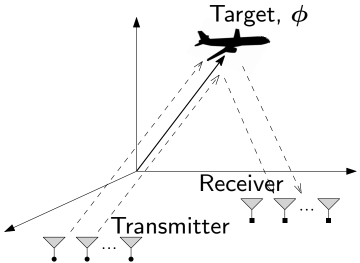
- Advances in robust and misspecified statistics[3] in the presence of *dependent data* are used to dispose of M3.

[3] H. White and I. Domowitz, "Nonlinear regression with dependent observations," *Econometrica*, vol. 52, no. 1, pp. 143–161, 1984.

Co-located MIMO system model

- We need radar systems with a large number N of spatial DoF: **co-located MIMO radars**

  - M_T transmitting antennas,
  - M_R receiving antennas,
  - $N \triangleq M_T M_R$: *virtual spatial antenna channels*.



- Signal collected at the receiving array:

$$\mathbf{x}(t) = \bar{\alpha}\mathbf{a}_R(\bar{\phi})\mathbf{a}_T^T(\bar{\phi})\mathbf{s}(t-\bar{\tau})e^{j\bar{\omega}t} + \mathbf{n}(t), \; t \in [0, T]$$

  - $\mathbf{a}_T(\phi) \in \mathbb{C}^{M_T}$: transmitting steering vector,
  - $\mathbf{a}_R(\phi) \in \mathbb{C}^{M_R}$: receiving steering vector.

Continuous-time signal model

$$\mathbf{x}(t) = \bar{\alpha}\mathbf{a}_R(\bar{\phi})\mathbf{a}_T^T(\bar{\phi})\mathbf{s}(t-\bar{\tau})e^{j\bar{\omega}t} + \mathbf{n}(t), \ t \in [0, T]$$

- $\mathbf{x}(t) \in \mathbb{C}^{M_R}$: array output vector at time t,
- $\mathbb{C}^{M_T} \ni \mathbf{s}(t) \triangleq \mathbf{W}\mathbf{s}_o(t)$: vector of transmitted signals
  - $\mathbf{W} \in \mathbb{C}^{M_T \times M_T}$ is the waveforms weighting matrix,
  - $\mathbf{s}_o(t)$: vector of *nearly* orthonormal signals,
- $\mathbf{n}(t) \in \mathbb{C}^{M_R}$: complex disturbance random process, or *clutter*.
- $\bar{\alpha} \in \mathbb{C}$ accounts for the target RCS the two-way path loss.

Co-located MIMO radar

$\bar{\alpha}$ is the same for each transmitter and receiver pair.

Discrete-time signal model (1/2)

- The output matrix $\mathbf{X}(l,k)$ of the filter matched to $\mathbf{s}_o(t)$ is[4]:

$$\mathbb{C}^{M_R \times M_T} \ni \mathbf{X}(l,k) = \bar{\alpha}\mathbf{a}_R(\bar{\phi})\mathbf{a}_T(\bar{\phi})^T \mathbf{W}\mathbf{S}(l,k) + \mathbf{C}(l,k),$$

- "Straddling loss" matrix:

$$\mathbf{S}(l,k) \triangleq \int_0^T \mathbf{s}_o(t-\bar{\tau})\mathbf{s}_o^H(t-l\Delta t)e^{-j(k\Delta\omega-\bar{\omega})t}dt$$

- Disturbance matrix:

$$\mathbf{C}(l,k) \triangleq \int_0^T \mathbf{n}(t)\mathbf{s}_o^H(t-l\Delta t)e^{-jk\Delta\omega t}dt.$$

- The range-Doppler indices (l,k) will be omitted next.

[4] B. Friedlander, "On signal models for MIMO radar," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 48, no. 4, pp. 3655–3660, October 2012.

Discrete-time signal model (1/2)

- The output matrix $\mathbf{X}$ can be expressed as:

$$\mathbb{C}^N \ni \mathbf{x} = \mathrm{vec}\left(\mathbf{X}\right) = \bar{\alpha}\mathbf{v}(\bar{\phi}) + \mathbf{c}$$

where $\mathbf{c} \triangleq \mathrm{vec}\left(\mathbf{C}\right)$ and:

$$\mathbf{v}(\bar{\phi}) = (\mathbf{S}^T \otimes \mathbf{I}_{M_R}) \left[\mathbf{W}^T \mathbf{a}_T(\bar{\phi}) \otimes \mathbf{a}_R(\bar{\phi})\right].$$

- If $\mathbf{n}(t)$ is a wide-sense stationary process, we have:

$$E\{\mathbf{n}(t)\} = \mathbf{0}, \forall t \quad \Rightarrow \quad E\{\mathbf{c}\} = \mathbf{0}$$

$$E\{\mathbf{n}(t)\mathbf{n}(\tau)^H\} = \mathbf{\Sigma}(t-\tau) \quad \Rightarrow \quad \mathbf{\Gamma} \triangleq E\{\mathbf{c}\mathbf{c}^H\}$$

$$\mathbf{\Gamma} = \iint \left[\mathbf{s}_o^*(t - l\Delta t)\mathbf{s}_o^T(t - l\Delta t) \otimes \mathbf{\Sigma}(t-\tau)\right] e^{-jk\Delta\omega(t-\tau)} dt d\tau.$$

Fully uncorrelated disturbance model

- Assumptions on the clutter process $\mathbf{n}(t)$:
  1. $\mathbf{n}(t)$ is spatially uncorrelated (along the receiving array),
  2. $\mathbf{n}(t)$ is also temporally uncorrelated (along T),

$$E\{\mathbf{n}(t)\mathbf{n}(\tau)^H\} = \mathbf{\Sigma}(t-\tau) = \sigma^2 \mathbf{I}_{M_R}\delta(t-\tau).$$

- If *perfect orthogonality* of the waveforms in $\mathbf{s}_o(t)$ is assumed:

$$\mathbf{\Gamma} = \sigma^2 \mathbf{I}_N = \begin{pmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma^2 \end{pmatrix}.$$

- This is a simple but unrealistic model.

Temporally uncorrelated disturbance model

- Assumption on the clutter process $\mathbf{n}(t)$:
  1. $\mathbf{n}(t)$ is also temporally uncorrelated (along T),

$$E\{\mathbf{n}(t)\mathbf{n}(\tau)^H\} = \boldsymbol{\Sigma}(t-\tau) = \boldsymbol{\Sigma}_R \delta(t-\tau).$$

- If *perfect orthogonality* of the waveforms in $\mathbf{s}_o(t)$ is assumed:

$$\boldsymbol{\Gamma} = \mathbf{I}_{M_T} \otimes \boldsymbol{\Sigma}_R = \begin{pmatrix} \boldsymbol{\Sigma}_R & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_R & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\Sigma}_R \end{pmatrix}.$$

- This is a more complex but still unrealistic model.

A more general disturbance models (1/2)

- We drop the too stringent[5,6] assumptions on
  - the temporal uncorrelation of $\mathbf{n}(t)$,
  - the perfect orthogonality of $\mathbf{s}_o(t)$,

  in favour of a much weaker requirement.

Our assumption

$[\boldsymbol{\Gamma}]_{i,j}$ goes to zero at least polynomially fast as $|i-j|$ increases.

- Moreover, unlike most of the existing literature, we do not require $\mathbf{c}$ to be Gaussian-distributed.

[5] B. Friedlander, "On spatial processing in MIMO radar," *ASILOMAR*, Pacific Grove, CA, 2011, pp. 2075-2079.

[6] B. Friedlander, "On signal models for MIMO radar," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 48, no. 4, pp. 3655–3660, October 2012.

A more general disturbance models (2/2)

- More formally, let $\mathbf{c} = [c_1, \ldots, c_N]^T$ be the disturbance vector.
- The entries $\{c_n\}_{n=1}^N$ can be considered as random variables sampled form a stationary discrete-time process $\{c_n : \forall n\}$.

Assumption A1: The autocorrelation function (ACF) of $\{c_n : \forall n\}$ satisfies

$$r_C[m] \triangleq E\{c_n c_{n-m}^*\} = O(|m|^{-\gamma})$$

where $m \in \mathbb{Z}$, $\gamma > \varrho/(\varrho - 1)$, $\varrho > 1$. [7]

- Note that we are not assuming any particular pdf p_C for $\mathbf{c}$, that will be left unspecified!

[7] H. White and I. Domowitz, "Nonlinear regression with dependent observations," *Econometrica*, vol. 52, no. 1, pp. 143–161, 1984.

ARMA disturbance model

- A stable $\mathrm{ARMA}(p,q)$ process, with finite p and q, satisfies Assumption A1 since its ACF decays exponentially fast.
- The second-order statistics of any discrete-time process with *continuous* Power Spectra Density (PSD) can be well-approximated by an ARMA model[8].
- A subset of the general ARMA models are the autoregressive model of order p, $\mathrm{AR}(p)$.
- AR models share most of the properties of the ARMA models.

[8] J. Li and P. Stoica, *MIMO Radar Signal Processing*. Hoboken, NJ: Wiley, 2009.

Autoregressive disturbance model

- A stable *stationary* AR(p) process $\{c_n : \forall n\}$ is a discrete random process s.t.:

$$c_n = \sum_{i=1}^{p} \rho_i c_{n-i} + w_n, \quad n \in (-\infty, \infty).$$

- The innovations w_n are zero-mean, *circularly symmetric*, i.i.d. random variables with $E\{|w_n|^2\} = \sigma_w^2 < \infty$.

- The pdf of w_n, say $p_W(w;\varphi)$ is generally non-Gaussian and may depends on an additional unknown *nuisance* vector φ.

- The ACF of a AR(p) process decays exponentially fast, so its satisfy Assumption A1.

Compound Gaussian (CG) disturbance model

- Any CG-distributed vector $\mathbf{c}$ admits a representation:

$$\mathbf{c} =_d \sqrt{\tau}\mathbf{s},$$

where:

  - the *texture* τ is a positive random variable,
  - the *speckle* $\mathbf{s} \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Gamma})$ is a complex Gaussian random vector with scatter/covariance matrix $\mathbf{\Gamma}$.

- The entries $\{s_n\}_{n=1}^{N}$ of the speckle can be considered as samples of a Gaussian AR(p) $\{s_v : \forall n\}$ with ACF $r_S[m]$.

- The speckle scatter matrix is then given by $[\mathbf{\Gamma}]_{i,j} = r_S[i-j]$, with $1 \leq i, j \leq N$.

- It is immediate to verify that the CG model satisfy A1.

On the pdf of the disturbance vector c

- Under AR(p) assumption, the pdf of **c** is given by:

$$p_C(\mathbf{c}; \boldsymbol{\rho}, \varphi) = g(\mathbf{c}; \boldsymbol{\rho}, \varphi) \prod_{n=p+1}^{N} p_W\left(r_n(\mathbf{c}, \boldsymbol{\rho}); \varphi\right),$$

  where:

  - $r_n(\mathbf{c}, \boldsymbol{\rho}) \triangleq c_n - \sum_{i=1}^{p} \rho_i c_{n-i}$,
  - $g(\mathbf{c}; \boldsymbol{\rho}, \varphi)$ is a function of the distribution of $\{c_{-p+1}, \ldots, c_0\}$.

- The disturbance pdf p_C depends on:

  - the pdf p_W of the innovations w_n,
  - the order p,

  that are both (generally) *unknown* in practice.

- Consequently, a classical GLRT *cannot* be implemented.

A robust HT problem

- Let us recall the MIMO detection problem:

$$
\begin{aligned}
H_0 &: \quad \mathbf{x} = \mathbf{c} \\
H_1 &: \quad \mathbf{x} = \bar{\alpha}\mathbf{v} + \mathbf{c},
\end{aligned}
$$

where:

- $\mathbf{v} \equiv \mathbf{v}(\phi) = (\mathbf{S}^T \otimes \mathbf{I}_{M_R}) \left[\mathbf{W}^T \mathbf{a}_T(\phi) \otimes \mathbf{a}_R(\phi)\right]$ is a known steering vector,
- $\bar{\alpha}$ is a deterministic unknown,
- $\mathbf{c}$ is the disturbance vector that is assumed to satisfy Assumption A1 but its true pdf p_C is unknown.

Final goal

Find a robust decision statistic whose asymptotic (as $N \to \infty$) distribution under H_0 does not depend on the *unknown* disturbance pdf p_C.

Main results: estimation

- The Least Square (LS) estimator of $\bar{\alpha}$ is $\hat{\alpha} = \mathbf{v}^H \mathbf{x} / ||\mathbf{v}||^2$.

Theorem 1

Under Assumption A1, the LS estimator $\hat{\alpha}$ is: [9]

1. *Consistent*: $\hat{\alpha} \underset{N \to \infty}{\overset{p}{\to}} \bar{\alpha}$,
2. Asymptotically normal: $\sqrt{N} \bar{B}_N^{-1/2} A_N (\hat{\alpha} - \bar{\alpha}) \underset{N \to \infty}{\sim} \mathcal{CN}(0, 1)$,

$$A_N \triangleq N^{-1} ||\mathbf{v}||^2, \quad \bar{B}_N \triangleq N^{-1} \mathbf{v}^H \boldsymbol{\Gamma} \mathbf{v}, \quad \boldsymbol{\Gamma} \triangleq E_{p_C}\{\mathbf{c}\mathbf{c}^H\},$$

with p_C being the unknown disturbance pdf.

[9] H. White and I. Domowitz, "Nonlinear regression with dependent observations," *Econometrica*, vol. 52, no. 1, pp. 143–161, 1984.

A consistent estimator for $\bar{B}_N$ (1/2)

- The scalar $\bar{B}_N$ is function of the unknown disturbance covariance matrix $\mathbf{\Gamma}$.

- A consistent estimator $\hat{B}_N$ of $\bar{B}_N$ is:

$$\hat{B}_N \equiv \hat{B}_N(\hat{\alpha}) = N^{-1}\mathbf{v}^H\widehat{\mathbf{\Gamma}}_l\mathbf{v},$$

where

$$[\widehat{\mathbf{\Gamma}}_l]_{i,j} \triangleq \begin{cases} \hat{c}_i\hat{c}_j^* & 0 \leq j-i \leq l \\ \hat{c}_i^*\hat{c}_j & 0 \leq i-j \leq l \\ 0 & |i-j| > l \end{cases} \quad 1 \leq i,j \leq N,$$

$$\hat{c}_n = x_n - \hat{\alpha}v_n,\ \forall n \quad \hat{\alpha} = \mathbf{v}^H\mathbf{x}/||\mathbf{v}||^2,$$

and l is the so-called *truncation lag*.

A consistent estimator for $\bar{B}_N$ (2/2)

Theorem 2

Under Assumption A1, if $l \to \infty$ as $N \to \infty$ such that $l = o(N^{1/3})$ then: [10]

$$\hat{B}_N - \bar{B}_N \underset{N\to\infty}{\overset{p}{\to}} 0.$$

- Theorems 1 and 2 tell us that, irrespective of the unknown p_C, the LS estimator $\hat{\alpha}$ is:
  - $\sqrt{N}$-consistent,
  - asymptotically normal estimator with asymptotic error covariance matrix given by $A_N^{-1}\bar{B}_N$,
  - a consistent estimate of $\bar{B}_N$ is provided by $\hat{B}_N$.
- A Wald-type test can be implemented!

[10] H. White and I. Domowitz, "Nonlinear regression with dependent observations," *Econometrica*, vol. 52, no. 1, pp. 143–161, 1984.

Main results: detection

A robust Wald-type test

The asymptotic characterization of the LS estimator leads to: [11]

$$\Lambda_{\text{RW}}(\mathbf{x}) = \frac{2N|\hat{\alpha}|^2}{A_N^{-2}\hat{B}_N} = \frac{2|\mathbf{v}^H\mathbf{x}|^2}{\mathbf{v}^H\widehat{\boldsymbol{\Gamma}}_l\mathbf{v}}.$$

Theorem 3

If Assumption A1 holds true, then:

$$\Lambda_{\text{RW}}(\mathbf{x}|H_0) \underset{N\to\infty}{\sim} \chi_2^2(0)$$
$$\Lambda_{\text{RW}}(\mathbf{x}|H_1) \underset{N\to\infty}{\sim} \chi_2^2(\varsigma)$$

where $\varsigma \triangleq 2|\bar{\alpha}|^2 \frac{||\mathbf{v}||^4}{\mathbf{v}^H\boldsymbol{\Gamma}\mathbf{v}}$.

[11] S. Fortunati, L. Sanguinetti, F. Gini, M. S. Greco, and M. Himed, "Massive MIMO radar for target detection", submitted to *IEEE Transactions on Signal Processing*, 2019 (https://arxiv.org/abs/1906.06191).

On the non-centrality parameter ς

- An explicit expression for ς is given by:

$$\varsigma = \frac{2|\bar{\alpha}|^2 M_R^2 \|(\mathbf{WS})^T \mathbf{a}_T(\bar{\phi})\|^4}{\mathrm{tr}\left(\boldsymbol{\Gamma}\left[(\mathbf{WS})^T \mathbf{a}_T(\bar{\phi})\mathbf{a}_T^H(\bar{\phi})(\mathbf{WS})^* \otimes \mathbf{a}_R(\bar{\phi})\mathbf{a}_R^H(\bar{\phi})\right]\right)}.$$

- By substituting $\boldsymbol{\Gamma}$ with its definition, we get:

$$\varsigma = \frac{2|\bar{\alpha}|^2 M_R \|(\mathbf{WS})^T \mathbf{a}_T(\bar{\phi})\|^2}{\iint_0^T ||\mathbf{s}_o(t-\bar{l}\Delta t)||^2 \mathrm{tr}\left[\boldsymbol{\Sigma}(t-\tau)\right] e^{-j\bar{k}\Delta\omega(t-\tau)} dt d\tau}.$$

- If $\mathbf{S} = \mathbf{I}_{M_T}$ and $\boldsymbol{\Sigma}(t-\tau) = \sigma^2 \mathbf{I}_{M_R}\delta(t-\tau)$: [12]

$$\varsigma = \frac{2|\bar{\alpha}|^2 P(\bar{\phi})}{\sigma^2}, \quad P(\bar{\phi}) \triangleq \mathbf{a}_T^H(\bar{\phi})\mathbf{W}^*\mathbf{W}^T \mathbf{a}_T(\bar{\phi}),$$

where $P(\bar{\phi})$ is the transmitting beam pattern.

[12] I. Bekkerman and J. Tabrikian, "Target detection and localization using MIMO radars and sonars," *IEEE Transactions on Signal Processing*, vol. 54, no. 10, pp. 3873–3883, Oct 2006

Asymptotic performance of $\Lambda_{\mathrm{RW}}(\mathbf{x})$

CFAR property and ROC curve

Under A1, the P_{FA} of $\Lambda_{\mathrm{RW}}(\mathbf{x})$ is asymptotically given by:

$$P_{FA} \to_{N\to\infty} e^{-\lambda/2},$$

irrespective of the unknown disturbance pdf p_C. Moreover,

$$P_D(P_{FA}) \to_{N\to\infty} Q_1\left(\frac{\sqrt{2}|\bar{\alpha}|||\mathbf{v}||^2}{\sqrt{\mathbf{v}^H\boldsymbol{\Gamma}\mathbf{v}}}, \sqrt{-2\ln P_{FA}}\right),$$

where $Q_1(\cdot,\cdot)$ is the Marcum Q function of order 1

- The minimum number N of virtual spatial DoF needed to well-approximate the asymptotic performance defines the **massive MIMO regime**.

A comparison with the AMF $\Lambda_{\text{AMF}}(\mathbf{x})$ [13]

$$\Lambda_{\text{AMF}}(\mathbf{x}) = \frac{|\mathbf{v}^H \widehat{\mathbf{C}}^{-1}\mathbf{x}|^2}{\mathbf{v}^H \widehat{\mathbf{C}}^{-1}\mathbf{v}}, \quad \Lambda_{\text{RW}}(\mathbf{x}) = \frac{2|\mathbf{v}^H \mathbf{x}|^2}{\mathbf{v}^H \widehat{\mathbf{\Gamma}}_l \mathbf{v}}.$$

- Multi-snapshots vs. Single-snapshot
  - $\Lambda_{\text{AMF}}(\mathbf{x})$ requires a set of homogeneous secondary snapshots to get the full rank estimation $\widehat{\mathbf{C}}$ of $\mathbf{\Gamma}$,
  - $\Lambda_{\text{RW}}(\mathbf{x})$ relies on a single spatial snapshot.
- Gaussian-based vs. Robust
  - $\Lambda_{\text{AMF}}(\mathbf{x})$ is a CFAR detector only if $\mathbf{c}$ and the set of secondary data are Gaussian-distributed,
  - $\Lambda_{\text{RW}}(\mathbf{x})$ is asymptotically CFAR for every disturbance vector $\mathbf{c}$ satisfying Assumption A1.

[13] F. C. Robey, D. R. Fuhrmann, E. J. Kelly, and R. Nitzberg, "A CFAR adaptive matched filter detector," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 28, no. 1, pp. 208–216, Jan 1992.

Numerical validation

We consider two different scenarios:

- Case 1: The disturbance is modelled as an AR(3) with

$$\bar{\boldsymbol{\rho}} = [0.5e^{j2\pi 0}, 0.3e^{-j2\pi 0.1}, 0.4e^{j2\pi 0.01}]^T,$$

- Case 2: The disturbance is modelled as an AR(6) with

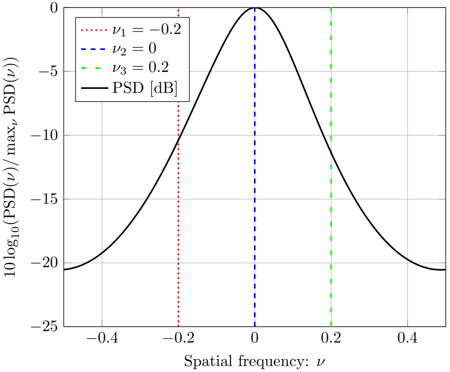
$$\bar{\boldsymbol{\rho}} = [0.5e^{-j2\pi 0.4}, 0.6e^{-j2\pi 0.2}, 0.7e^{j2\pi 0}, 0.4e^{j2\pi 0.1}, \\ 0.5e^{j2\pi 0.3}, 0.6e^{j2\pi 0.35}]^T,$$

- In both cases, the innovations $\{w_n, \forall n\}$ share a complex t-distribution:

$$p_w(w_n; \lambda, \eta) = (\sigma_w^2 \pi)^{-1} \lambda (\lambda/\eta)^{\lambda} (\lambda/\eta + |w_n|/\sigma_w^2)^{-(\lambda+1)}$$

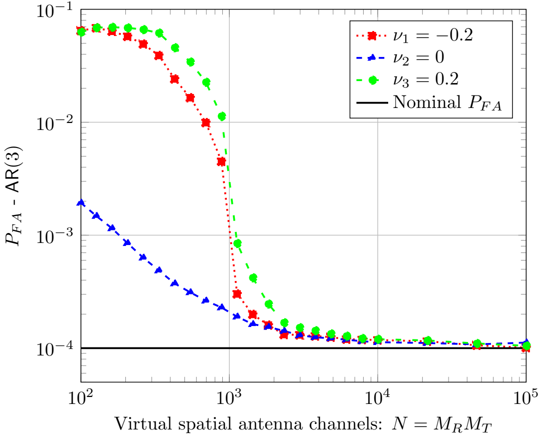
where $\lambda = 2$, $\sigma_w^2 = 1$ and $\eta = \lambda/\sigma^2(\lambda - 1)$.

Power Spectral Density (PSD) of the AR(3)



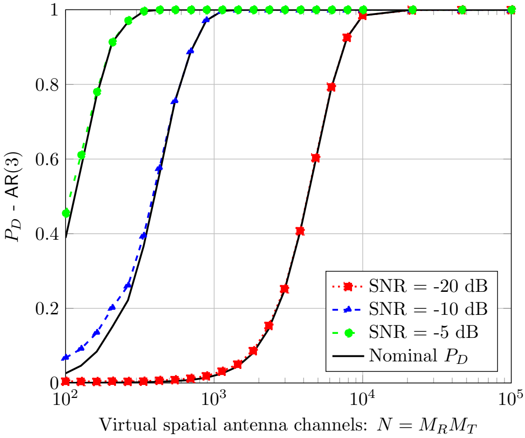
- Virtual steering vectors: $[\mathbf{v}_i]_n = e^{j\pi(n-1)\sin(\phi_i)}$, $n = 1, \ldots, N$ and $\phi_i = \arcsin(\nu_i/2)$ where $\nu_1 = -0.2$, $\nu_2 = 0$ and $\nu_3 = 0.2$.
- $\mathbf{S} = \mathbf{I}_{M_T}$ and $\mathbf{W} = \mathbf{I}_{M_T}$.

Estimated and theoretical P_{FA}: case 1



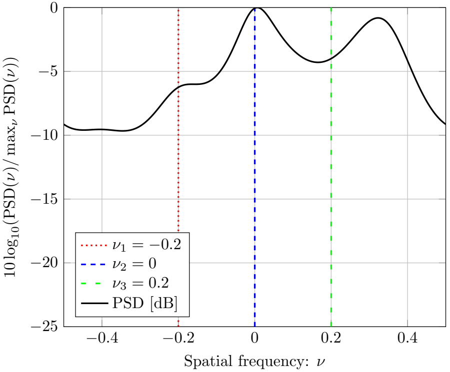
- The estimated P_{FA} tends to the nominal value $\overline{P_{FA}} = 10^{-4}$,
- The massive MIMO regime is achieved for $N = M_R M_T \geq 10^4$.

Estimated and theoretical P_D: case 1



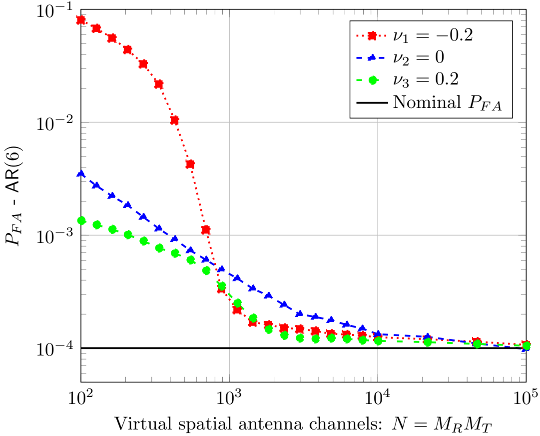
- The SNR is defined as $\mathrm{SNR} \triangleq 10 \log_{10}(|\bar{\alpha}|^2/\sigma^2)$.
- The estimated P_D is close to the asymptotic approximation.

Power Spectral Density (PSD) of the AR(6)



- Virtual steering vectors: $[\mathbf{v}_i]_n = e^{j\pi(n-1)\sin(\phi_i)}$, $n = 1, \ldots, N$ and $\phi_i = \arcsin(\nu_i/2)$ where $\nu_1 = -0.2$, $\nu_2 = 0$ and $\nu_3 = 0.2$.
- $\mathbf{S} = \mathbf{I}_{M_T}$ and $\mathbf{W} = \mathbf{I}_{M_T}$.

Estimated and theoretical P_{FA}: case 2



- The estimated P_{FA} tends to the nominal value $\overline{P_{FA}} = 10^{-4}$,
- The massive MIMO regime is achieved for $N = M_R M_T \geq 10^4$.

Concluding remarks

- By exploiting the increased number N of spatial DoF that a co-located MIMO radar can provide, a robust Wald-type detector Λ_{RW} is proposed.
- As $N = M_R M_T \to \infty$ and if the disturbance ACF decays at least polynomially fast, the asymptomatic distribution of Λ_{RW} does not depend on the *unknown* disturbance pdf.
- This represents a first attempt to apply the "massive" MIMO paradigm of communication systems to radar applications.
- Analogy:
  - imperfect knowledge of propagation channels in communication systems,
  - imperfect knowledge of the disturbance pdf in radar systems.

Open issues and future works

- The convergence to the nominal P_{FA} is quite slow, i.e. it requires a (maybe too) large number of virtual spatial DoF $N = M_R M_T$.
- *Possible solution*: additional structure to the disturbance statistical model $\rightarrow$ **Semiparametric approach**.
- An $ARMA(p, q)$ model may be assumed with:
  - a finite-dim. nuisance vector of the pq coefficients,
  - an infinite-dim. nuisance, i.e. the pdf of the innovation.
- The role of the waveform matrix $\mathbf{W}$ has not been explored yet.
- Using the **reinforcement learning**, $\mathbf{W}$ may be iteratively optimized to maximize the P_D (through ς). [14]

[14] L. Wang, S. Fortunati, M. S. Greco and F. Gini "Reinforcement Learning-based waveform optimization for MIMO multi-target detection," *Asilomar*, Pacific Grove, CA, USA, Oct. 28-31, 2018.