Trajectory Based Approach for the Stability Analysis of Nonlinear Systems with Time Delays

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***Abstract*—We propose a new technique for stability analysis for nonlinear dynamical systems with delays and possible discontinuities. In contrast with Lyapunov based approaches, our trajectory based approach involves verifying certain inequalities along solutions of auxiliary systems. It applies to a wide range of systems, notably time-varying systems with time-varying delay, ODE coupled with difference equations, and networked control systems with delay. It relies on the input-to-state stability notion, and yields input-to-state stability with respect to uncertainty.**

Key Words: nonlinear, delay, continuous-discrete, stability, input-to-state stability.

I. Introduction

Stabilizing nonlinear systems with delays is important because of latencies in communication and other factors. Much of the delays literature is based on Lyapunov-Krasovskii functionals or Razhumikin functions, which apply to coupled delay differential and difference equations [24] and hybrid systems describing networked control systems (NCS) [2] in continuous or discrete time [15]. This has led to a literature on building Lyapunov functions or Lyapunov-Krasovskii functionals for nonlinear delay systems; see [3], [4], [7], [17], [19], and [20]. Much of the literature involves transforming strict Lyapunov functions for the systems obtained by setting all of the delays to zero, into Lyapunov-Krasovskii functionals or Razhumikin functions [19].

However, finding strict Lyapunov functions can be difficult. Our work [16] gave methods for building strict Lyapunov functions for many ODE and discrete time systems, but those methods may not always apply. Instead, it may be easier to check that there are constants $\rho \in (0,1)$ and $\mathfrak{T} > 0$ such that every solution ζ of a system satisfies an inequality of the type $|\zeta(t)| \leq \rho \sup_{l\in[t-\mathfrak{T},t]} |\zeta(l)|$, especially for neutral systems (as described in [1], [6], [17], [25] and Remark 7 below), delay systems that are of continuous-discrete type, and ODE coupled with difference or integral equations [12], [18], [23]. See [26] for ISS for continuous time systems, [8] for ISS for impulsive systems, and [10] and [14] for discrete time ISS. See also [27] for sufficient conditions for ISS under time delays and certain gain conditions, using Razumikhin functions.

To help overcome these challenges, this note presents a new trajectory based approach, without building Lyapunov functionals. We view some terms (which frequently contain delays) as disturbing terms, and our conditions ensure that they do not destabilize the dynamics. We develop this method for dynamics with time-varying pointwise delay, ODE coupled with difference equations, and delayed NCS [21], [22], [28], including neutral systems. While [1] was devoted to the fundamental families of linear neutral systems, the present work applies to an interesting class of nonlinear neutral systems, and it leads to ISS estimates for some neutral systems; see Remark 7. All results we establish are global, but we can also state local versions.

Two possible advantages of our approach are that (a) it covers a significant range of time-varying nonlinear systems with arbitrarily long time-varying delays and (b) our methods may be easier to apply than existing Lyapunov or small-gain methods and give explicit comparison functions in ISS estimates with exponential decay rates. See, e.g., Remark 2 on the potential advantages of our trajectory approach, over the small gain and Lyapunov-based approaches for interconnections of ODEs and static systems from [9], [12]. Interconnected systems often have transfer phenomena that lead to delays [5], [29]. The NCS structure is widely used, but is often degraded by delayed communication [2]. The novelty of our NCS treatment is that it leads to sufficient conditions for ISS under arbitrarily long delays, and our results on interconnected systems are novel because they do not require Lyapunov functions for the subsystems.

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II. Notation and Key Lemma

Let $n \in \mathbb{N}$ be arbitrary, and $|\cdot|$ be the Euclidean 2 norm of matrices or vectors. For any measurable essentially bounded $\mathbb{R}^n$ valued function ϕ having an interval $\mathcal{I}$ in its domain, let $|\phi|_{\mathcal{I}}$ be its essential supremum over $\mathcal{I}$. A function $\mathcal{F}$ defined on $\mathcal{I}$ is called piecewise continuous provided (a) there are at most finitely many points on each bounded subinterval of $\mathcal{I}$ where $\mathcal{F}$ is discontinuous and (b) its one sided limits exist and are finite at each point in $\mathcal{I}$. We use the classes $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ of comparison functions [16].

Take any time-varying system of the form $\dot{x}(t) = \mathcal{G}(t, x(t), \delta(t))$, with the state space $\mathbb{R}^N$ and measurable essentially bounded perturbations $\delta : [0,+\infty) \to \mathbb{R}^M$, for any N and M. The system is called input-to-state stable (ISS) (with respect to δ valued in $\mathbb{R}^M$) [13] provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for each choice of δ, each initial time $t_0 \geq 0$, and each initial function x_0, the corresponding unique trajectory $x(t)$ is defined on $[t_0,+\infty)$ and satisfies $|x(t)| \leq \beta(|x_0|_{\mathcal{I}}, t-t_0) + \gamma(|\delta|_{[t_0,t]})$ for all $t \geq t_0$. The special case where there are no perturbations and the γ term in the ISS estimate is not present is called uniform global asymptotic stability (UGAS). We also use UGAS to mean uniformly globally asymptotically stable. We set $w(t^-) = \lim_{s\to t^-} w(s)$ and $w(t^+) = \lim_{s\to t^+} w(s)$, when these one sided limits exist. We also use $\mathrm{ceiling}(r)$ to denote the smallest integer $J \geq r$ for each $r \in \mathbb{R}$. We prove the following key lemma in the appendix (but see Remark 1 for a discussion on the assumptions of this lemma):

Lemma 1. *Let $T^* > 0$ be a constant. Let a piecewise continuous function $w : [-T^*,+\infty) \to [0,+\infty)$ admit a sequence of real numbers v_i and positive constants $\overline{v}_a$ and $\overline{v}_b$ such that $v_0 = 0$, $v_{i+i} - v_i \in [\overline{v}_a, \overline{v}_b]$ for all $i \geq 0$, w is continuous on each interval $[v_i, v_{i+1})$ for all $i \geq 0$, and $w(v_i^-)$ exists and is finite for each $i \in \mathbb{N}$. Let $d : [0,+\infty) \to [0,+\infty)$ be any piecewise continuous function, and assume that there is a constant $\rho \in (0,1)$ such that*

$$w(t) \leq \rho |w|_{[t-T^*,t]} + d(t) \tag{1}$$

holds for all $t \geq 0$. Then

$$w(t) \ \leq \ |w|_{[-T^*,0]} e^{\frac{\ln(\rho)}{T^*} t} + \frac{1}{(1-\rho)^2} |d|_{[0,t]} \tag{2}$$

holds for all $t \geq 0$. □

III. Introductory Result

We study the class of all nonlinear systems of the form

$$\dot{x}(t) \ = \ f\big(t, x(t), \zeta(t,\tau), \delta(t)\big) \tag{3}$$

with piecewise continuous perturbations $\delta : [0,+\infty) \to \mathbb{R}^m$ for any dimension m, where $\zeta(t,\tau) = (X_1(t-\tau_1(t)), X_2(t-\tau_2(t)), \ldots, X_L(t-\tau_L(t)))$, each subvector X_i of x has some dimension n_i for $i = 1, 2, \ldots, L$ such that $n_1 + n_2 + \ldots + n_L = n$, $\tau = (\tau_1, \tau_2, \ldots, \tau_L)$ is a vector of piecewise continuous pointwise time-varying delays, f is locally Lipschitz, x is valued in $\mathbb{R}^n$, $\tau_i(t) \in [\tau_S, \tau_M]$ for all t and i, and τ_M and τ_S are any constants satisfying $\tau_M \geq \tau_S > 0$, but see Remark 2 for generalizations. We use capital letters for the subvectors of x, to distinguish them from scalar components. The initial functions are defined on $[t_0 - \tau_M, t_0]$ where t_0 is the arbitrary initial time, and n is arbitrary. We assume:

Assumption 1. *The system*

$$\dot{\xi}(t) = f(t, \xi(t), u(t)) \tag{4}$$

is ISS with respect to u valued in $\mathbb{R}^n \times \mathbb{R}^m$ and ξ valued in $\mathbb{R}^n$. □

Assumption 1 provides functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $t_0 \geq 0$ and all $t \geq t_0$, the inequality

$$|\xi(t)| \quad \leq \quad \beta\big(|\xi(t_0)|, t - t_0\big) + \gamma\left(|u|_{[t_0,t]}\right) \tag{5}$$

holds along all trajectories of (4) for all piecewise continuous u's that are valued in $\mathbb{R}^{n+m}$. We assume that β and γ satisfy:

Assumption 2. *There are constants $T \geq \tau_M$ and $\rho_0 \in (0,1)$ such that*

$$\alpha(s) = \beta(s, T) + \gamma(s) \tag{6}$$

satisfies

$$\alpha(s) \leq \rho_0 s \text{ for all } s > 0 \tag{7}$$

where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ satisfy the ISS requirements from Assumption 1. □

In particular, this means that $\gamma(s) < s$ for all $s > 0$. To prove our first theorem, we also need the following:

Lemma 2. *Let $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$, and the constants $T > 0$, $\tau_S > 0$, and $\tau_M > 0$ satisfy the preceding requirements, and define the sequence $\mathcal{L}_i \in \mathcal{K}_\infty$ by*

$$\mathcal{L}_0(r) = \beta(r, 0) + \gamma(2r) + r \tag{8}$$

and

$$\mathcal{L}_{j+1}(r) = \mathcal{L}_0(2\mathcal{L}_j(r)) + \gamma(2r) \tag{9}$$

for all integers $j \geq 0$. Set $\bar{v} = \text{ceiling}(\max\{0, (2T/\tau_S) - 1\})$. Then

$$|x(t)| \ \leq \ \mathcal{L}_{\bar{v}}(|x|_{[t_0-\tau_M,t_0]})e^{2T+t_0-t} + \mathcal{L}_{\bar{v}}(|\delta|_{[t_0,t]}) \tag{10}$$

holds for all solutions $x(t)$ of (3), all choices of δ, all initial times $t_0 \geq 0$, and all $t \in [t_0, t_0 + 2T]$.

Proof. Choose any solution $x(t)$ of (3) for any choices of δ and $t_0 \geq 0$. We use induction to prove that for all integers $i \geq 0$ and all $t \in [t_0 - \tau_M, t_0 + (i+1)\tau_S]$, we have

$$|x(t)| \leq \mathcal{L}_i(|x|_{[t_0-\tau_M,t_0]}) + \mathcal{L}_i(|\delta|_{[t_0,t]}). \tag{11}$$

The lemma will then follow because $t_0 + (\bar{v}+1)\tau_S \geq t_0 + 2T$. For all $t \in [t_0 - \tau_M, t_0 + \tau_S]$, our Assumption 1 gives

$$\begin{aligned} |x(t)| \quad &\leq \quad \beta(|x(t_0)|, 0) + \gamma(|x|_{[t_0-\tau_M,t_0]} + |\delta|_{[t_0,t]}) \\ &\qquad + |x|_{[t_0-\tau_M,t_0]} \\ &\leq \quad \beta(|x(t_0)|, 0) + \gamma(2|x|_{[t_0-\tau_M,t_0]}) \\ &\qquad + \gamma(2|\delta|_{[t_0,t]}) + |x|_{[t_0-\tau_M,t_0]}, \end{aligned} \tag{12}$$

which gives the result for $i = 0$. Assume that the induction assumption holds for some index $i \geq 0$. Then for each $t \in [t_0 + (i+1)\tau_S, t_0 + (i+2)\tau_S]$, we get

$$\begin{aligned} |x(t)| \quad &\leq \quad \beta(|x(t_0 + (i+1)\tau_S)|, 0) \\ &\qquad + \gamma(|x|_{[t_0-\tau_M,t_0+(i+1)\tau_S]} + |\delta|_{[t_0,t]}) \\ &\leq \quad \mathcal{L}_0(|x|_{[t_0-\tau_M,t_0+(i+1)\tau_S]}) + \gamma(2|\delta|_{[t_0,t]}) \\ &\leq \quad \mathcal{L}_0(2\mathcal{L}_i(|x|_{[t_0-\tau_M,t_0]})) + \gamma(2|\delta|_{[t_0,t]}) \\ &\qquad + \mathcal{L}_0(2\mathcal{L}_i(|\delta|_{[t_0,t]})), \end{aligned} \tag{13}$$

so the induction hypothesis (11) holds for $i+1$. Hence, the lemma follows by induction. □

We use Lemmas 1-2 to prove:

Theorem 1. *Let Assumptions 1-2 hold, where u is valued in $\mathbb{R}^n \times \mathbb{R}^m$. Then (3) is ISS with respect to the set of all piecewise continuous perturbations δ. Moreover, with the choices*

$$\begin{aligned} &\overline{\beta}(s,t) = \mathcal{L}_{\bar{v}}(s)\left(e^{\frac{\ln(\rho_0)}{2T}(t-2T)} + e^{2T-t}\right) \\ &\text{and } \ \overline{\gamma}(s) = \tfrac{s\rho_0}{(1-\rho_0)^2} + \mathcal{L}_{\bar{v}}(s)\ , \end{aligned} \tag{14}$$

the ISS estimate $|x(t)| \leq \overline{\beta}(|x|_{[t_0-\tau_M,t_0]}, t-t_0) + \overline{\gamma}(|\delta|_{[t_0,t]})$ holds along all trajectories of (3) for all times $t \geq t_0 \geq 0$ and all δ. □

Proof. Assumption 1 gives

$$\begin{aligned} |x(t)| \quad &\leq \quad \beta\left(|x|_{[t-2T,t]} + |\delta|_{[t_0,t]}, T\right) \\ &\qquad + \gamma\left(|x|_{[t-2T,t]} + |\delta|_{[t_0,t]}\right) \\ &\leq \quad \rho_0\left(|x|_{[t-2T,t]} + |\delta|_{[t_0,t]}\right) \end{aligned} \tag{15}$$

along all trajectories of (3) for all $t \geq t_0 + 2T$ and all piecewise continuous functions δ (by using the initial time $t - T$ in Assumption 1). We apply Lemma 1 with $w(t) = |x(t + 2T + t_0)|$, $\rho = \rho_0$, $T_* = 2T$, and $d(t) = \rho_0|\delta|_{[t_0,t_0+2T+t]}$ to get

$$|x(t)| \ \leq \ |x|_{[t_0,t_0+2T]}e^{\frac{\ln(\rho_0)}{2T}(t-2T-t_0)} + \tfrac{\rho_0}{(1-\rho_0)^2}|\delta|_{[t_0,t]} \tag{16}$$

for all $t \geq 2T + t_0$. Also, Lemma 2 gives

$$|x|_{[t_0,t_0+2T]} \ \leq \ \mathcal{L}_{\bar{v}}(|x|_{[t_0-\tau_M,t_0]}) + \mathcal{L}_{\bar{v}}(|\delta|_{[t_0,t_0+2T]}). \tag{17}$$

The choices (14) of $\overline{\beta}$ and $\overline{\gamma}$ then follow by substituting (17) into (16), and then checking separately that $|x(t)| \leq \overline{\beta}(|x|_{[t_0-\tau_M,t_0]}, t - t_0) + \overline{\gamma}(|\delta|_{[t_0,t]})$ is satisfied for all $t \in [t_0, t_0 + 2T]$ and then for all $t \geq t_0 + 2T$. The fact that the final ISS estimate holds for all $t \in [t_0, t_0 + 2T]$ follows as a consequence of (10). □

Remark 1. *There are many continuous functions w satisfying (1). For example, since d is nonnegative, this is the case for $w(t) = e^{-\epsilon t}$ for any $\epsilon \geq -\ln(\rho)/T^*$, $\rho \in (0,1)$, and $T_* > 0$. Lemma 1 is no longer true if we replace the requirement that (1) holds by the weaker requirement that there is a continuous function λ such that $0 < \lambda(s) < s$ for all $s > 0$ and such that*

$$w(t) \leq \lambda\left(|w|_{[t-T^*,t]}\right) + d(t) \tag{18}$$

holds for all $t \geq 0$. To see why, choose the function $\lambda(s) = s - \frac{s^2}{1+s^2}$ and any constant $T^* > 0$. Then (18) becomes

$$w(t) \leq |w|_{[t-T^*,t]} - \frac{|w^2|_{[t-T^*,t]}}{1+|w^2|_{[t-T^*,t]}} + d(t)\ . \tag{19}$$

Therefore, if w is nondecreasing, then we have $|w|_{[t-T^*,t]} = w(t)$, and then the inequality (19) with the choice $d \equiv 1$ becomes

$$w(t) \leq w(t) - \frac{w^2(t)}{1 + w^2(t)} + 1 \tag{20}$$

The preceding inequality is always satisfied, so we have solutions that violate any a priori given ISS inequality. To overcome this obstacle, Lemma 1 requires the constant $\rho \in (0,1)$. We conclude that Lemma 1 can perhaps be proven under a less restrictive assumption, but the assumption that such a ρ exists cannot be removed without being replaced by another assumption. □

Remark 2. *The function $(\zeta(t,\tau), \delta(t))$ in (3) is valued in $\mathbb{R}^n \times \mathbb{R}^m$, so it is valid to replace it by an $\mathbb{R}^{n+m}$ valued disturbance u. Our results remain true if we drop the local Lipschitzness assumption, provided we understand the ISS assumption to hold for all trajectories for all initial conditions, to allow cases where the uniqueness of* *solutions properties may not hold. This covers differential inclusions and discontinuous dynamics. Another approach to (3) is to rewrite it as an interconnection $\dot{x}(t) = f(t, x(t), z(t))$, $z(t) = \zeta(t, \tau)$ of an ODE and a static system. The work [12] proves general stability results for interconnections of this type, and [9] builds Lyapunov-Krasovskii functionals for important special cases of the interconnections in [12]. Like many small gain results, we do not need regularity conditions such as Lipschitzness that are assumed in [12]; and we do not require formulas for Lyapunov functions for the subsystems, which are needed for [9]. However, three potential advantages of our approach are that: (a) it often suffices to check that our estimates along trajectories hold for large times (instead of all times starting from the initial times) without needing to compare the norm of the current state to the norm of the initial function (as we demonstrate in (30) below), (b) we do not require growth conditions such as $\tau_i'(t) \in [0,1]$ on the delay components (which are often required in the delay literature when $t - \tau_i(t)$ is a lower limit of integration in a Lyapunov-Krasovskii functional), and (c) our results provide closed form expressions for the comparison functions in exponential ISS results (as we did in (14), in terms of the function $\mathcal{L}_{\bar{v}}$ from Lemma 2), while small gain results generally conclude UGAS or ISS without finding explicit comparison functions.* □

Remark 3. *We can extend our work to systems of the form $\dot{x}(t) = f(t, x(t), \zeta(t, \tilde{\tau}), \zeta(t, \tau))$ where τ and $\tilde{\tau}$ can be different vectors of delays, under an ISS assumption on $\dot{x}(t) = f(t, x(t), \zeta(t, \tilde{\tau}), u(t))$. Our assumptions allow sampling with delay, by defining each τ_i by $\tau_i(t) = t - t_{j-1}$ for all $t \in [t_j, t_{j+1})$ and $j \in \mathbb{N}$ and $\tau_i(t) = t_1 - t_0$ for all $t \in [t_0, t_1)$, and assuming that the sequence $0 = t_0 < t_1 < t_2 < \dots$ of sampling times admits constants $\bar{t}_S > 0$ and $\bar{t}_M > 0$ such that $\bar{t}_S \le t_j - t_{j-1} \le \bar{t}_M$ for all $j \ge 1$.* □

Remark 4. *In many cases, including linear time-varying systems, one can choose β and γ of the form $\gamma(s) = b_1 s$ and $\beta(s,t) = b_2 s e^{-b_3 t}$ for some constants $b_i > 0$. If $b_1 \ge 1$, then Assumption 2 is not satisfied. If $b_1 < 1$, and if we choose $T = \max\{0, \ln(2b_2/(1-b_1))/b_3\}$, then (6) becomes $\alpha(l) = \frac{1+b_1}{2} l$, so Assumption 2 holds. Some systems such that $b_1 \ge 1$ are not asymptotically stable, such as $\dot{x}(t) = -x(t) + x(t-\tau)$, which admits all constant solutions. Thus, it does not seem possible to relax Assumption 2.* □

Remark 5. *There are always many possible choices for β and γ in the ISS estimate (5). To satisfy Assumption 2, one should look for the smallest possible choices, to minimize conservativeness. See [30] for methods for computing tight integral ISS estimates and related results for ISS. Since our results ensure UGAS or ISS, our assumptions imply the existence of Lyapunov-Krasovskii functionals for our systems. However, our work is helpful because it eliminates the need for searching for Lyapunov functionals to conclude UGAS or ISS. Note too that the only restrictions we need on the piecewise continuous functions τ_i are that $\tau_S \le \tau_i(t) \le \tau_M$ for all i and t.* □

IV. ODE Coupled with a Difference Equation

In the rest of this paper, we limit our analysis to systems without perturbations δ and with constant input delays, but we can use Lemma 1 to prove analogous results for perturbed cases with time-varying delays. We next consider the time-varying system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f\big(t, x(t), z(t-\tau)\big) \\ z(t) & = & \epsilon z(t-\tau) + g\big(t, x(t)\big), \end{array} \right. \tag{21}$$

where x is valued in $\mathbb{R}^n$, z is valued in $\mathbb{R}^p$, f and g are two locally Lipschitz nonlinear functions, and $\tau \ge 0$ and ϵ are constants; see Remark 7 for the motivation for studying the structure (21). The dimensions n and p are arbitrary. We introduce a set of assumptions:

Assumption 3. *The system*

$$\dot{\xi}(t) = f\big(t, \xi(t), u(t)\big) \tag{22}$$

is ISS with respect to u valued in $\mathbb{R}^p$.

Assumption 4. *There is a constant $c > 0$ such that the inequality*

$$\big|g(t,x)\big| \le c|x| \tag{23}$$

is satisfied for all $x \in \mathbb{R}^n$ and $t \ge 0$.

Assumption 3 provides functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $t_0 \ge 0$ and all $t \ge t_0$, the ISS inequality (5) is satisfied along all trajectories of (22). We assume that β and γ are known and satisfy:

Assumption 5. *There are a constant $T \ge \tau$ and a constant $\rho \in (0,1)$ such that the function $\alpha(s) = \beta(s,T) + \gamma(s)$ satisfies*

$$\alpha(s) \le \rho s \ \text{ for all } \ s > 0 \ . \tag{24}$$

Moreover, the constant

$$r = \frac{-[|\epsilon| + c\rho - \rho] + \sqrt{(|\epsilon| + c\rho - \rho)^2 + 4c\rho^2}}{2c\rho} \tag{25}$$

is such that

$$|\epsilon| + c\rho\,(r+1) < 1 \tag{26}$$

holds.

Assumption 5 is stronger than simply requiring $|\epsilon| < 1$, because it requires $1 - |\epsilon| > c\rho$ and because we cannot reduce the linear growth rate on $\alpha(s) = \beta(s,T) + \gamma(s)$. Note for later use that

$$\rho\,(1 + 1/r) = |\epsilon| + c\rho(r+1). \tag{27}$$

Our main result for (21) is as follows:

Theorem 2. *If Assumptions 3-5 hold, then (21) is UGAS.* □

Proof. Consider any solution of the system (21). Then for all times $t_0 \ge 0$ and all $t \ge t_0$ such that the solution is defined up to time t, we can use Assumptions 3 and 4 to get

$$\begin{array}{rcl} |x(t)| & \le & \beta(|x(t_0)|, t - t_0) + \gamma\left(|z|_{[t_0 - \tau, t-\tau]}\right) \\ |z(t)| & \le & |\epsilon||z(t-\tau)| + c|x(t)| \ . \end{array} \tag{28}$$

By successively considering the intervals $[t_0 + (i-1)\tau, t_0 + i\tau]$ for all $i \in \mathbb{N}$, we deduce that no finite escape time is possible. Hence, all of the maximal solutions are defined over $[t_0, +\infty)$ for all initial times $t_0 \ge 0$. Moreover, by successively considering the intervals $[t_0 + i\tau, t_0 + (i+1)\tau]$ for integers $i \ge 0$ and using Assumption 3 and the form of the z subsystem in (21), we can find a function $\gamma_0 \in \mathcal{K}_\infty$ that is independent of the initial time $t_0 \ge 0$ such that $|(x(t), z(t))| \le \gamma_0(|(x,z)|_{[t_0-\tau, t_0]}) e^{2T + t_0 - t}$ for all $t \in [t_0, t_0 + 2T]$.

It follows that for all $t \ge T + t_0$, we have

$$\begin{array}{rcl} |x(t)| & \le & \beta\big(|x(t-T)|, T\big) + \gamma\left(|z|_{[t-T-\tau, t-\tau]}\right) \\ |z(t)| & \le & |\epsilon||z(t-\tau)| + c|x(t)| \ . \end{array} \tag{29}$$

From the definition (6) of α, we get the following for all $t \ge 2T + t_0$:

$$\begin{array}{rcl} |x(t)| & \le & \alpha\left(|x|_{[t-T-\tau, t]} + |z|_{[t-T-\tau, t]}\right) \\ |z(t)| & \le & |\epsilon||z|_{[t-T-\tau, t]} \\ & & + c\alpha\left(|x|_{[t-T-\tau, t]} + |z|_{[t-T-\tau, t]}\right) \ . \end{array} \tag{30}$$

We define the function

$$\widetilde{w}(t) = \max\{|x(t)|, r|z(t)|\}, \tag{31}$$

where $r > 0$ is the constant (25) in Assumption 5. Notice that (30) implies that for all $t \ge 2T + t_0$, we have

$$\begin{array}{l} \widetilde{w}(t) \le \max\left\{\alpha\left(|x|_{[t-T-\tau,t]} + |z|_{[t-T-\tau,t]}\right),\right. \\ r|\epsilon||z|_{[t-T-\tau,t]} + rc\alpha\left(|x|_{[t-T-\tau,t]} + |z|_{[t-T-\tau,t]}\right)\left.\right\} \ . \end{array} \tag{32}$$

From (24) and (32), we get

$$\begin{array}{rcl}\widetilde{w}(t) & \leq & \max\left\{\alpha\left(|\widetilde{w}|_{[t-T-\tau,t]}+\frac{1}{r}|\widetilde{w}|_{[t-T-\tau,t]}\right),\right.\\ & & |\epsilon||\widetilde{w}|_{[t-T-\tau,t]}\\ & & \left.+rc\alpha\left(|\widetilde{w}|_{[t-T-\tau,t]}+\frac{1}{r}|\widetilde{w}|_{[t-T-\tau,t]}\right)\right\}\\ & \leq & \overline{\alpha}\left(|\widetilde{w}|_{[t-T-\tau,t]}\right)\end{array} \tag{33}$$

for all $t \geq 2T + t_0$, where

$$\begin{array}{rcl}\overline{\alpha}(l) & = & \max\left\{\rho\left(1+\frac{1}{r}\right)l, |\epsilon|l + rc\rho\left(1+\frac{1}{r}\right)l\right\}\\ & = & \max\left\{\rho\left(1+\frac{1}{r}\right), |\epsilon| + c\rho\left(r+1\right)\right\}l\\ & = & \left(|\epsilon|+c\rho\left(r+1\right)\right)l\end{array} \tag{34}$$

and the last equality followed from (27). The theorem then follows from (26) and Lemma 1, by taking $w(l) = \widetilde{w}(l + 2T + t_0)$, the constant $|\epsilon| + c\rho(r + 1) \in (0, 1)$, and $T^* = T + \tau$ to get a UGAS estimate that applies for all $t \geq 2T + t_0$, since we already found a UGAS estimate that applies for values $t \in [t_0, t_0 + 2T]$. □

Remark 6. *The requirement (26) is satisfied in a reasonably large number of cases. It is equivalent to*

$$|\epsilon| + c\rho + \rho + \sqrt{(|\epsilon| + c\rho - \rho)^2 + 4c\rho^2} < 2 \tag{35}$$

which is satisfied if, for instance, $c = 1$, $|\epsilon| = \rho$ and $\rho < 2/(3+\sqrt{5})$. □

Remark 7. *Theorem 2 makes it possible to analyze the stability of neutral systems of the form*

$$\dot{z}(t) - \epsilon\dot{z}(t-\tau) = b\big(t, z(t)\big)\ , \tag{36}$$

where z is valued in $\mathbb{R}^n$ and $\tau > 0$ and ϵ are constants. Indeed, let $x(t) = z(t) - \epsilon z(t-\tau)$. Then we obtain

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & b\big(t, x(t) + \epsilon z(t-\tau)\big)\\ z(t) & = & \epsilon z(t-\tau) + x(t)\end{array}\right. \tag{37}$$

which has the form (21). See [11, Chap. 1] for more motivation for studying systems of the type (21). □

V. NCS WITH DELAY

We next use Lemma 1 to establish UGAS results for NCS having a constant time delay $\tau > 0$. We define the sequence $t_k = kT$ for all nonnegative integers k and all constants $T > 0$. To simplify the analysis, we assume that $T = \tau$. Our NCS is defined by

$$\left\{\begin{array}{l}\left\{\begin{array}{rcl}\dot{x}(t) & = & f\big(t, x(t), x(t-\tau), z(t)\big)\\ \dot{z}(t) & = & g\big(t, x(t), z(t)\big)\\ & & \forall t \in [t_k, t_{k+1}) \text{ and } k \in \{0, 1, 2, \ldots\}\end{array}\right.\\ \quad z(t_k) \;=\; h\big(z(t_k^-)\big),\ \ k \in \mathbb{N}\ ,\end{array}\right. \tag{38}$$

where x is valued in $\mathbb{R}^n$, z is valued in $\mathbb{R}^p$, and f, g and h are nonlinear locally Lipschitz functions. As before, the dimensions n and p are arbitrary. The system (38) represents an NCS with a pointwise delay in the continuous dynamics, for the reasons explained in [22, Section III]. For simplicity, we assume that the initial times for all trajectories of (38) are 0, and we denote the set of all initial functions for (38) by $C_{\rm in}$. However, to prove our result, we need to use arbitrary initial times in our assumptions:

Assumption 6. *There are constants $c_1 \geq 0$, $c_2 \geq 0$, and $c_3 > 0$ such that (a) for all $z \in \mathbb{R}^p$, the inequality*

$$|h(z)| \leq c_1|z| \tag{39}$$

is satisfied and (b) all solutions of the system

$$\dot{\xi}(t) = g\big(t, v(t), \xi(t)\big) \tag{40}$$

for all continuous functions v satisfy

$$|\xi(t)| \leq e^{c_3(t-t_0)}|\xi(t_0)| + c_2|v|_{[t_0,t]} \tag{41}$$

for all initial times $t_0 \geq 0$ and all $t \geq t_0$. Also, the system

$$\dot{\xi}(t) = f\big(t, \xi(t), u(t)\big) \tag{42}$$

is ISS with respect to $u \in \mathbb{R}^n \times \mathbb{R}^p$.

As before, Assumption 6 provides functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all initial times $t_0 \geq 0$ and all $t \geq t_0$, the ISS inequality (5) holds along all trajectories of (42). We assume that β and γ are known and satisfy:

Assumption 7. *There exist an integer $j \geq 1$ and a constant $\rho \in (0, 1)$ such that the function*

$$\alpha(s) = \beta(s, j\tau) + \gamma(s) \tag{43}$$

satisfies

$$\alpha(s) \leq \rho s \text{ for all } s \geq 0\ , \tag{44}$$

where β and γ are functions satisfying the ISS condition on (42).

Assumption 8. *The inequalities*

$$\rho(1 + c_2) < 1, \tag{45}$$

$$e^{c_3\tau}c_1 < 1, \tag{46}$$

and

$$c_1^j e^{jc_3\tau} + \rho + \sqrt{\left(c_1^j e^{jc_3\tau} - \rho\right)^2 + \frac{4\rho c_2\left(1 + c_1 e^{c_3\tau}\right)}{1 - c_1 e^{c_3\tau}}}\ <\ 2 \tag{47}$$

are satisfied.

Remark 8. *By (46), the contractive reset function h has a stabilizing effect. See [2] for more motivation for the model (38). Assumption 8 holds in a reasonably large number of cases. For instance, if $c_1 = 0$, then (47) reduces to the delay independent condition $\rho + \sqrt{\rho^2 + 4\rho c_2} < 1$.*

We are ready to state and prove the following result:

Theorem 3. *If (38) satisfies Assumptions 6-8, then it is UGAS for all initial conditions in $C_{\rm in}$.* □

Proof. Take any initial function for system (38) defined on $[-\tau, 0]$. (Since there is no delay in $z(t)$, we can assume that z is constant on $[-\tau, 0]$.) If $t \in [0, \tau]$ is any value such that the corresponding solution is defined over $[-\tau, t)$, then Assumptions 6 and 7 imply that

$$\begin{array}{rcl}\big|x(l)\big| & \leq & \beta(|x(0)|, l) + \gamma\left(\sup_{s\in[0,l]}\big|(x(s-\tau), z(s))\big|\right)\\ |z(l)| & \leq & e^{c_3 l}|z(0)| + c_2|x|_{[0,l]}.\end{array} \tag{48}$$

for all $l \in [0, t)$. It follows that

$$\begin{array}{rcl}\big|x(l)\big| & \leq & \beta(|x(0)|, l) + \gamma\left(\sup_{s\in[0,l]}|x(s-\tau)|\right.\\ & & \left.+e^{c_3 l}|z(0)| + c_2|x|_{[0,l]}\right).\end{array} \tag{49}$$

Since $\gamma(s) \leq \rho s$ for all $s \geq 0$, we deduce that

$$\begin{array}{rcl}\big|x(l)\big| & \leq & \beta(|x(0)|, l) + \rho\sup_{s\in[0,l]}|x(s-\tau)|\\ & & +\rho e^{c_3 l}|z(0)| + \rho c_2|x|_{[0,l]}\\ & \leq & \beta(|x(0)|, l) + \rho(1 + c_2)|x|_{[-\tau,l]}\\ & & +\rho e^{c_3 l}|z(0)|.\end{array} \tag{50}$$

Hence, $|x|_{[0,t]} \leq \beta(|x(0)|, 0) + \rho(1 + c_2)|x|_{[-\tau,t]} + \rho e^{c_3 t}|z(0)|$. It follows that

$$\begin{array}{l}[1 - \rho(1 + c_2)]|x|_{[0,t]}\\ \leq\ \beta(|x(0)|, 0) + \rho(1 + c_2)|x|_{[-\tau,0]} + \rho e^{c_3 t}|z(0)|.\end{array} \tag{51}$$

By (45), we deduce that the solution is defined over $[-\tau, \tau]$. Reasoning similarly over each interval $[k\tau, (k+1)\tau]$ for all $k \in \mathbb{N}$, we can prove that all of the maximal solutions are defined over $[-\tau, +\infty)$.

Assumption 6 implies that for all $k \in \mathbb{N}$, we have $|z(t_k)| \le c_1|z(t_k^-)|$. For all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}$, we have $t - \tau \in [t_{k-1}, t_k)$, so we deduce from Assumption 6 that

$$\begin{aligned} |z(t)| &\le e^{c_3(t-t_k)} c_1 |z(t_k^-)| + c_2|x|_{[t_k,t]} \\ &\le e^{c_3(t-t_k)} c_1 \big[e^{c_3(t_k - t+\tau)} |z(t-\tau)| \\ &\quad + c_2|x|_{[t-\tau,t_k]} \big] + c_2|x|_{[t_k,t]} \\ &\le c_1 e^{c_3\tau}|z(t-\tau)| + e^{c_3\tau} c_1 c_2 |x|_{[t-\tau,t_k]} + c_2|x|_{[t_k,t]} \\ &\le c_1 e^{c_3\tau}|z(t-\tau)| + (1 + c_1 e^{c_3\tau})\, c_2 |x|_{[t-\tau,t]}\ , \end{aligned}$$

where the second inequality followed by applying the inequality (41). Similar reasoning implies that for all $t \ge (j+1)\tau$, where $j \ge 1$ is from Assumption 7, and for all $i \in \{0, 1, \dots, j-1\}$, we have

$$\begin{aligned} |z(t-i\tau)| &\le c_1 e^{c_3\tau}|z(t-(i+1)\tau)| \\ &\quad + (1 + c_1 e^{c_3\tau})\, c_2 \sup_{l\in[t-(i+1)\tau, t-i\tau]} |x(l)|\ . \end{aligned} \tag{52}$$

Hence,

$$\begin{aligned} & c_1^i e^{ic_3\tau}|z(t-i\tau)| - c_1^{(i+1)} e^{(i+1)c_3\tau}|z(t-(i+1)\tau)| \\ & \le (e^{c_3\tau}c_1)^i (1 + c_1 e^{c_3\tau})\, c_2 \sup_{l\in[t-j\tau,t]} |x(l)| \end{aligned} \tag{53}$$

for all $t \ge (j+1)\tau$ and all $i \in \{0, 1, \dots, j-1\}$. Summing the terms in (53) over all $i \in \{0, 1, \dots, j-1\}$, we get

$$\begin{aligned} |x(t)| &\le \beta(|x(t-j\tau)|, j\tau) \\ &\quad + \gamma\left(|x|_{[t-(j+1)\tau, t-\tau]} + |z|_{[t-j\tau,t]}\right) \\ |z(t)| &\le c_1^j e^{jc_3\tau}|z|_{[t-j\tau,t]} + c_4|x|_{[t-(j+1)\tau,t]} \end{aligned} \tag{54}$$

for all $t \ge (j+1)\tau$, where

$$c_4 = \frac{(1 + c_1 e^{c_3\tau})\, c_2}{1 - c_1 e^{c_3\tau}} \tag{55}$$

and where we used the condition $e^{c_3\tau}c_1 < 1$ from Assumption 8 and a geometric sum to get the second line in (54). Also, Assumption 7 implies that for all $t \ge (j+1)\tau$, we have

$$|x(t)| \le \rho\left(|x|_{[t-(j+1)\tau,t]} + |z|_{[t-(j+1)\tau,t]}\right)\ . \tag{56}$$

Next note that the constant

$$r = \frac{1}{2c_4}\left(c_1^j e^{jc_3\tau} - \rho + \sqrt{\left(c_1^j e^{jc_3\tau} - \rho\right)^2 + 4\rho c_4}\right) \tag{57}$$

is such that

$$\begin{aligned} & \tfrac{1}{2}\left(c_1^j e^{jc_3\tau} + \rho + \sqrt{\left(c_1^j e^{jc_3\tau} - \rho\right)^2 + 4\rho c_4}\right) \\ & = \rho + rc_4 = c_1^j e^{jc_3\tau} + \tfrac{\rho}{r}\ . \end{aligned} \tag{58}$$

It follows from (54), (56), and (58) that for all $t \ge (j+1)\tau$,

$$\begin{aligned} |x(t)| + r|z(t)| &\le (\rho + rc_4)\, |x|_{[t-(j+1)\tau,t]} \\ &\quad + \left(c_1^j e^{jc_3\tau} + \tfrac{\rho}{r}\right) r|z|_{[t-(j+1)\tau,t]} \\ &\le (\rho + rc_4) \\ &\quad \times \sup_{l\in[t-(j+1)\tau,t]} (|x(l)| + r|z(l)|)\ . \end{aligned} \tag{59}$$

Then (47), (58) and Lemma 1 (applied with $w(t) = |x(t + (j+1)\tau)| + r|z(t + (j+1)\tau)|$, $T^* = (j+1)\tau$, $\delta = 0$, and the constant $\rho + rc_4 \in (0,1)$) give the necessary UGAS estimate that is valid for all times $t \ge (j+1)\tau$. Then we can argue as in the last part of the proof of Theorem 1 to get a UGAS estimate that is valid on $[0, (j+1)\tau]$. The final UGAS estimate then follows by adding the right sides of the UGAS estimate that is valid for all $t \in [0, (j+1)\tau]$ to the UGAS estimate that is valid for $t \ge (j+1)\tau$. □

Remark 9. *We conjecture that Theorem 3 can be extended to other families of instants of discontinuities than $t_k = k\tau$, where the differences $t_{k+1} - t_k$ between the sampling times do not need to be constant. For the sake of simplicity, we considered the case where the only delay is in f. We conjecture that analogous arguments can cover the case where there is a delay in g as well.* □

Remark 10. *While we only considered perturbations acting on the dynamics from Section III, analogous reasoning gives ISS results for the more complex systems in Sections IV and V. We leave the details to the reader. See [22] for motivation for studying NCS with uncertainties, and for stability results for NCS in the absence of delays in the vector fields. See also [17] for stability results for nonlinear neutral systems using Lyapunov functional constructions, leading to robustness estimates of the form*

$$|x(t)| \le \beta(|x(0)| + |\dot{x}|_{[-\tau,0]}, t) + \gamma\left(\int_0^t |\delta(s)|\mathrm{d}s\right) \tag{60}$$

or

$$|x(t)| \le \beta(|x(0)| + |\dot{x}|_{[-\tau,0]}, t) + \gamma(|\delta|_{[0,t]}) \tag{61}$$

for certain functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$. □

VI. Conclusions

We developed a new technique for the analysis of several families of nonlinear systems with time delays. It applies to systems with time-varying delays, ordinary differential equations coupled with difference equations, systems with delay of neutral type, and networked control systems. Instead of using standard methods such as Lyapunov-Krasovskii functionals or Razhumikhin functions, it is trajectory based, and can be applied with great simplicity. Since our approach is trajectory based, it can be applied with the usual regularity conditions on the dynamics, such as Lipschitzness, and can in fact allow discontinuous dynamics; see [14] for analysis of discrete time discontinuous systems based on Lyapunov methods. Moreover, our result can be used to design controllers for the families of systems we mentioned, by indicating properties for the controllers that give closed-loop systems that satisfy the assumptions of our main results.

Appendix: Proof of Lemma 1

We set $A(t) = |w|_{[t-T^*,t]}$ for all $t \ge 0$, $\Delta(t) = |d|_{[t-T^*,t]}$ for all $t \ge T^*$, $\beta = -\ln(\rho)/T^*$, and $B(t) = e^{\beta t}A(t)$ for all $t \ge 0$. Since $\rho \in (0,1)$, we have $\beta > 0$. Also, $e^{-\beta T^*} = \rho$.

Step 1. It follows from (1) that for all $t \ge 0$,

$$\begin{aligned} |w|_{[0,t]} &\le \rho|w|_{[-T^*,t]} + |d|_{[0,t]} \\ &= \rho \max\left\{|w|_{[-T^*,0]}, |w|_{[0,t]}\right\} + |d|_{[0,t]}. \end{aligned} \tag{A.1}$$

Next, we consider two cases. First case: $|w|_{[-T^*,0]} > |w|_{[0,t]}$. Then $|w|_{[0,t]} \le \rho|w|_{[-T^*,0]} + |d|_{[0,t]}$. Second case: $|w|_{[-T^*,0]} \le |w|_{[0,t]}$. Then $|w|_{[0,t]} \le \rho|w|_{[0,t]} + |d|_{[0,t]}$, which implies that $|w|_{[0,t]} \le |d|_{[0,t]}/(1-\rho)$. Therefore, in both cases, the inequality

$$|w|_{[0,t]} \le \rho A(0) + \frac{1}{1-\rho}|d|_{[0,t]} \tag{A.2}$$

is satisfied.

Step 2. From (1), we easily deduce that for all $t \ge T^*$, we have $A(t) \le \rho \max\{A(t-T^*), A(t)\} + \Delta(t)$. Fix any $t \ge T^*$. Next, we distinguish between two cases. First case: $A(t-T^*) > A(t)$. Then $A(t) \le \rho A(t-T^*) + \Delta(t)$. Second case: $A(t-T^*) \le A(t)$. Then $A(t) \le \rho A(t) + \Delta(t)$, so $A(t) \le \frac{1}{1-\rho}\Delta(t)$. Therefore, in both cases,

$$A(t) \le \rho A(t-T^*) + \frac{\Delta(t)}{1-\rho} \tag{A.3}$$

is satisfied. From the definition of $B(t) = e^{\beta t}A(t)$, it straightforwardly follows that

$$e^{-\beta t}B(t) \quad \leq \quad e^{-\beta(t-T^*)}\rho B(t-T^*) + \tfrac{1}{1-\rho}\Delta(t). \tag{A.4}$$

Consequently,

$$\begin{array}{rcl} B(t) & \leq & e^{\beta T^*}\rho B(t-T^*) + \frac{e^{\beta t}}{1-\rho}\Delta(t) \\ & = & B(t-T^*) + \frac{e^{\beta t}}{1-\rho}\Delta(t), \end{array} \tag{A.5}$$

where the last equality used the fact that $e^{-\beta T^*} = \rho$. Since $t \geq T^*$, there is an integer $J \geq 1$ such that $t \in [JT^*, (J+1)T^*)$. Then, we deduce from (A.5) that

$$\begin{array}{rcl} B(t) & \leq & B(t-JT^*) + \displaystyle\sum_{i=0}^{J-1} \frac{e^{\beta(t-iT^*)}}{1-\rho}\Delta\big(t-iT^*\big) \\ & \leq & B(t-JT^*) + \frac{e^{\beta t}}{1-\rho}\displaystyle\sum_{i=0}^{J-1} e^{-\beta iT^*}|d|_{[t-JT^*,t]} \\ & \leq & B(t-JT^*) + \frac{e^{\beta t}}{(1-\rho)^2}|d|_{[t-JT^*,t]} \ , \end{array} \tag{A.6}$$

where we used a geometric sum. This immediately gives

$$A(t) \ \leq \ e^{-J\beta T^*}A(t-JT^*) + \frac{1}{(1-\rho)^2}|d|_{[t-JT^*,t]}. \tag{A.7}$$

Since $t \in [JT^*, (J+1)T^*)$, it follows that

$$\begin{array}{rcl} A(t) & \leq & e^{-\beta t+\beta T^*}A(t-JT^*) + \frac{1}{(1-\rho)^2}|d|_{[t-JT^*,t]} \\ & \leq & e^{-\beta t+\beta T^*}|w|_{[-T^*,T^*]} + \frac{1}{(1-\rho)^2}|d|_{[t-JT^*,t]} \\ & = & e^{-\beta t+\beta T^*}\max\big\{A(0), A(T^*)\big\} \\ & & + \frac{1}{(1-\rho)^2}|d|_{[t-JT^*,t]}. \end{array} \tag{A.8}$$

Since $\rho \in [0,1)$, it follows from (A.2) that $A(T^*) \leq A(0) + \Delta(T^*)/(1-\rho)$. We deduce that

$$\begin{array}{rcl} A(t) & \leq & e^{-\beta t+\beta T^*}A(0) + \frac{\Delta(T^*)}{1-\rho}e^{-\beta t+\beta T^*} \\ & & + \frac{1}{(1-\rho)^2}|d|_{[t-JT^*,t]}. \end{array} \tag{A.9}$$

Since $t \geq JT^* \geq T^*$, it follows that

$$\begin{array}{rcl} A(t) & \leq & e^{-\beta t+\beta T^*}A(0) + \frac{1}{1-\rho}|d|_{[0,T^*]} \\ & & + \frac{1}{(1-\rho)^2}|d|_{[t-JT^*,t]} \\ & \leq & e^{-\beta t+\beta T^*}A(0) + \frac{2-\rho}{(1-\rho)^2}|d|_{[0,t]}. \end{array} \tag{A.10}$$

Step 3. From (1) and (A.10), we deduce that for all $t \geq T^*$,

$$\begin{array}{rcl} w(t) & \leq & \rho|w|_{[t-T^*,t]} + d(t) \\ & \leq & \rho e^{-\beta t+\beta T^*}A(0) + \frac{\rho(2-\rho)}{(1-\rho)^2}|d|_{[0,t]} + d(t). \end{array} \tag{A.11}$$

It follows from (A.2) that for all $t \geq 0$, we have

$$w(t) \ \leq \ \rho A(0)e^{-\beta t+\beta T^*} + \left(1 + \frac{\rho(2-\rho)}{(1-\rho)^2}\right)|d|_{[0,t]}.$$

Since $\rho e^{\beta T^*} = 1$, this completes the proof of the lemma.

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