Nonlinear Quantum Observer Design for Continuous Optical Phase Estimation

Continuous optical phase estimation [1] is a fundamental problem in quantum optics, precision metrology, and feedback control. In many quantum-optical systems, the phase carries the information of interest, yet it cannot be measured directly. Instead, it must be inferred in real time from noisy continuous measurement records generated through homodyne detection and adaptive feedback. This leads naturally to a nonlinear estimation problem in which the phase is time-varying, the measurement law depends nonlinearly on the phase mismatch, and the underlying dynamics arise from a quantum-optical setting. Developing observer design methods for this regime is therefore both practically important and theoretically significant.

While coherent quantum observers [2, 3] have already been developed, the nonlinear quantum-observer problem remains much less understood hence the central objective of this project is to develop a nonlinear quantum observer for continuous optical phase estimation. The project takes the phase-estimation problem not merely as an application example, but as a concrete and experimentally relevant setting in which a nonlinear quantum-observer framework can be formulated, analysed, and validated. The main goal is to identify an observer structure that captures the nonlinear nature of adaptive homodyne measurement while remaining sufficiently tractable for mathematical analysis and meaningful implementation.

The motivation for this direction comes from the limitations of existing estimation approaches. Standard linear filters and smoothers are typically derived from linearised models and perform well only when the phase mismatch remains small. However, the homodyne photocurrent depends nonlinearly on the phase error, usually through a sinusoidal relation, so the validity of linearised estimation is inherently local. Under larger uncertainty, stronger transients, or increased measurement noise, the nonlinear structure of the measurement becomes important and may no longer be well captured by linear approximation. Moreover, smoothing-based methods are acausal and therefore unsuitable for real-time feedback and online estimation.

The first direction of the project is to formulate a nonlinear state-space description of continuous optical phase estimation that is suitable for observer design. Following experimentally relevant models, the phase may be described by a resonant stochastic process driven by white noise, while the measurement is obtained from adaptive homodyne detection. A representative measurement model [4] is

$$I(t) = 2|\alpha| \sin\big(\phi(t) - \hat{\phi}(t)\big) + dW(t), \tag{1}$$

where $\phi(t)$ denotes the true optical phase, $\hat{\phi}(t)$ is an intermediate estimate used within the adaptive measurement loop, $|\alpha|$ is the coherent amplitude, and $dW(t)$ represents measurement noise. A key task is to reformulate this phase-measurement interaction into a nonlinear stochastic model that clearly exposes the system dynamics, the measurement mechanism, and the estimation error structure in a form appropriate for observer synthesis.

The second direction is the construction of a nonlinear quantum observer architecture for real-time phase tracking. The aim here is not simply to apply standard nonlinear filtering tools to a quantum-optical example, but to determine what kind of observer structure is appropriate when the measured signal is generated by a continuous quantum measurement process and embedded in feedback. In this project, the term nonlinear quantum observer refers to an observer framework designed for a quantum dynamical setting, with explicit attention to quantum measurement structure, physical interpretability, and possible pathways toward implementation, rather than a purely classical estimator applied to quantum-generated data.

The third direction is the analysis of the resulting observer. For a nonlinear observer of this type, numerical performance alone is not sufficient. It is essential to understand whether the observer is well posed, whether the estimation error remains bounded, and under what conditions one may expect convergence or local stability. Particular emphasis will be placed on robustness with respect to modelling uncertainty, measurement noise, and operating regimes in which the phase mismatch is not small. A central goal is to move beyond simulation-based intuition and toward a principled observer design framework for this class of nonlinear quantum estimation problems.

A fourth direction concerns the relation between estimation and feedback in adaptive quantum-optical measurement. In continuous optical phase estimation, the observer is not merely a passive processor of measurement data; the evolving estimate also influences the measurement configuration through the feedback loop. This coupling suggests that observer design and feedback design may need to be understood together. Studying this interaction is expected to shed light on how nonlinear quantum observers should be formulated in settings where measurement, estimation, and control are intrinsically linked.

From a methodological perspective, the project begins with a resonant phase model of the form

$$G(s) = \frac{\phi(s)}{v(s)} = \frac{\kappa}{s^2 + 2\zeta\omega_r s + \omega_r^2}, \tag{2}$$

where κ is the gain, ζ is the damping ratio, ω_r is the resonant frequency, and v is a white-noise input. This process model will be combined with the nonlinear homodyne measurement equation to obtain a nonlinear continuous-time estimation problem. On this basis, the project will derive a state-space representation, construct candidate observer equations, and investigate the associated error dynamics. Numerical studies with realistic optical parameters will then be used to compare the proposed nonlinear quantum-observer framework with existing linearised estimators in terms of mean-square error, transient response, and sensitivity to uncertainty.

The expected contribution of this project is therefore twofold. At the theoretical level, it seeks to formulate and analyse a nonlinear quantum observer problem arising from continuous optical phase estimation, thereby creating a bridge between nonlinear observer theory and continuous quantum measurement. At the practical level [5], it aims to identify observer structures that can improve real-time phase tracking in regimes where the nonlinear measurement relation cannot be neglected. More broadly, the project is intended to position continuous optical phase estimation as a tractable and physically meaningful testbed for the development of nonlinear quantum observer theory.

References

[1] S. Roy, O. U. Rehman, I. R. Petersen, and E. H. Huntington, “Robust smoothing for estimating optical phase varying as a continuous resonant process,” in *Proceedings of the European Control Conference 2014*, 2014, p. 896–901.

[2] S. L. Vuglar and N. Amini, “Design of coherent quantum observers for linear quantum systems,” *New Journal of Physics*, vol. 16, no. 12, p. 125005, 2014.

[3] J. E. Gough and N. H. Amini, “How to build a quantum coherent observer,” in *2018 IEEE 57th Conference on Decision and Control (CDC)*, 2018, p. 6597–6601.

[4] M. Collett, R. Loudon, and C. Gardiner, “Quantum theory of optical homodyne and heterodyne detection,” *Journal of Modern Optics*, vol. 34, no. 6–7, p. 881–902, 1987.

[5] A. Serafini, “Feedback control in quantum optics: An overview of experimental breakthroughs and areas of application,” *ISRN Optics*, vol. 2012, p. 1–15, 2012.